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STATISTICAL THEORY AND METHOD

COVERAGE OF JOURNAL

THE aim of this journal of abstracts is to give complete coverage of papers in the field of statistical theory and new contributions to statistical method. Papers which report only applications or examples of existing statistical theory and method will not be included. There are approximately two hundred and fifty journals published in various parts of the world which are wholly or partly devoted to the field of statistical theory and method and which will be brought within the scope of this journal of abstracts. A complete list of journals covered is printed in the annual Index Supplement. In the case of the following journals, however, being those which are wholly devoted to statistical theory—all contributions, whether a paper, note or miscellanea, will be abstracted:

Annals of Mathematical Statistics
Biometrika
Journal, Royal Statistical Society (Series B)
Bulletin of Mathematical Statistics
Annals, Institute of Statistical Mathematics

Within the larger group of journals, which are not wholly devoted to statistical theory and method, there are some journals which have the vast majority of their contributions in this field. These journals, therefore, will be abstracted on a virtually complete basis:

Biometrics
Metrika
Metron
Review, International Statistical Institute
Technometrics
Sankhyā

After experience in the publication of this journal of abstracts it may be found desirable to add further journals to these lists. In any case readers may be assured that all papers properly to be included in this journal of abstracts will be included irrespective of such notification. If any reader of this journal discovers a paper which happens to have been overlooked, the General Editor will be pleased to be informed so that the appropriate abstract can be made: always provided that the date of publication is after 1st October 1958, when the abstracting for this journal commenced.

In addition to the ordinary journals, there are two kinds of publication which fall within the scope of this journal of abstracts. They are the experiment and other research station reports—which occur particularly in the North American region—and the reports of conferences, symposia and seminars. Whilst these latter may be included in the book review sections of journals it is unusual for any individual contribution to be noted at any length. These publications are, in effect, special collections of papers and for this reason the appropriate arrangements will be made for them to be included in this journal. By the same token, abstracts of papers given at conferences and reproduced in an appropriate journal will be disregarded until the definitive publication is available.

FORMAT OF JOURNAL

The abstracts will be up to 400 words long—the recommendation of UNESCO for the "long" abstract service: they will be in the English language although the language of the original paper will be indicated on the abstract together with the name of the abstractor. In addition, the address of the author(s) will be given in sufficient detail to facilitate contact in order to obtain further detail or request an offprint. Suitable indexes will be provided annually in a supplement.

The scheme of classification has been developed upon lines that will facilitate the transfer to punched cards of the code numbers allocated to each abstract: to allow for future development it is suggested that use is made of 4-column fields. Each abstract will have two classification numbers: the primary number in heavy type to indicate the basic topic of the paper and the secondary number in brackets to take account of the most important cross-reference. The sheets of the journal are colour-coded according to the twelve main sections of the classification and it should be noted that it is the main section number of the primary classification which determines the colour code for each abstract. It is believed that this method of colour-coding the pages will provide a distinctive visual aid to the identification of abstracts both when the journal is in bound form or dismantled and filed on cards or in binders. The format and simple binding allows of the following alternative treatments by users of the journal:

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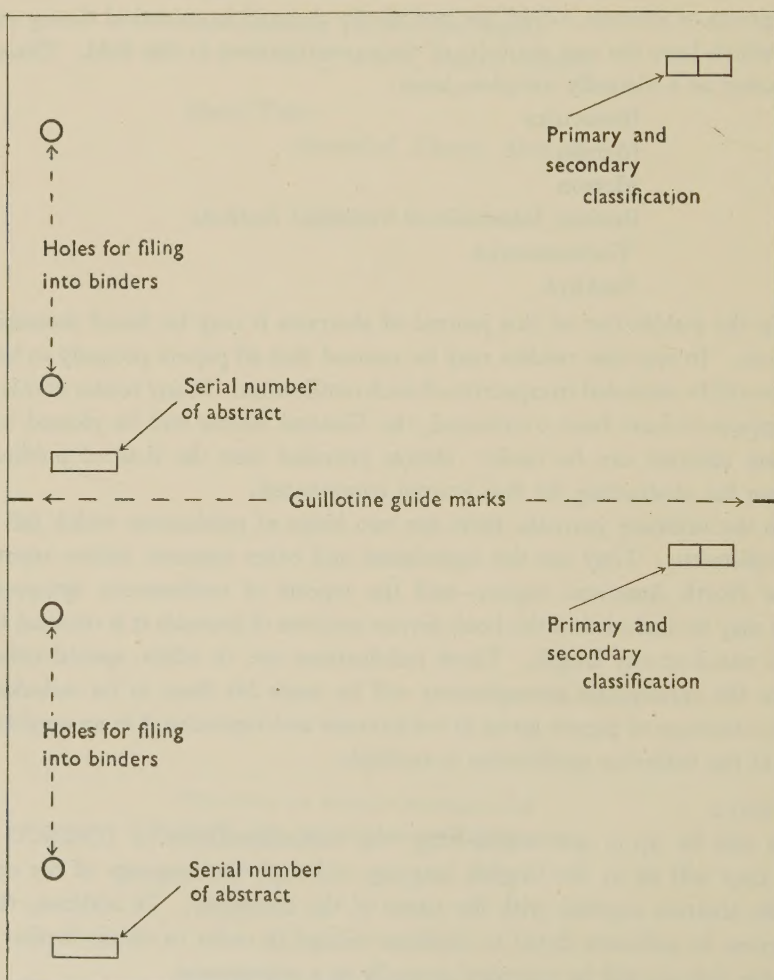

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JOURNAL OF THE AMERICAN STATISTICAL ASSOCIATION

VOLUME 56 — JUNE 1961 — NUMBER 294

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Volume 1. CORRECTIONS

1/544	**1.5**	Frankx	Author's name should read " Franckx ".
1/566	**2.9**	Bahadur	Third line from end: last word should read " generalisation ".
1/588	**3.3**	Ruben	Name in 6th line should read " Behrens-Fisher ".
1/680	**8.0**	Mikulski & others	Title of paper should read " . . . mean properties . . . ". Line 5 should read $\bar{w}$. Line 15, insert word " if " before $w(r)$.
1/683	**8.0**	Oderfeld	Character at end of 1st line should read $\bar{w}$.
1/749	**11.1**	Federigh	Author's name should read " Federighi ".
Author Index No. 4 and in Supplement			Amend author's name to " Federighi ".

Volume 2. CORRECTIONS

2/121	**8.9**	Barnard	The secondary classification should read (10.2).
2/131	**8.8**	Lieberman & Bowker	The expression in col. 2, line 5 should read $1-f$.
2/213	**10.1**	Steinhaus & Urbanik	Column 2, line 11 should read $\langle 0, 1 \rangle$.
2/219	**11.1**	Keuls	The journal abstracted is *Statist Neerlandica.*
Author Index No. 1			Add: Barnard, G. A. **8.9, 8.9**; Durbin, J. **10.2**; Kiefer, J. **9.0, 9.0.**

EDITORIAL NOTE: In this and future issues all serial numbers of abstracts, whether at the bottom left-hand corner of an item, quoted as a reference or mentioned in a Corrections slip, will show the volume number of this Journal in bold face: e.g., **1**/462.

ARNAIZ, V. G. & LASUEN, J. R. (Inst. Nacional de Estadística, Madrid) **0.6** (-.-)

Economic significance of the coefficients in input-output analysis—*In Spanish*

Estadística Española (1959) 5, 27-33.

The economic significance of all the inversion process in Leontief's input-output matrix is explained from a didactical point of view. The authors emphasise the importance of the matrix $I-A$, not only as a necessary intermediate step between A and $(I-A)^{-1}$, but also as a meaningful element in itself; whose interpretation is not so obvious because of its two kinds of coefficients. A deeper comprehension of each algorithm is urged as an instrument for research and its applications.

(F. Azorín)

2/225

BARTON, D. E. & MERRINGTON, Maxine (University College, London) **0.1** (2.5)

Tables for the solution of the exponential equation—*In English*

Biometrika (1960) **47**, 439-446 (10 references, 2 tables)

The authors consider the solution of the exponential equation $\exp(-a)+ka = 1$ for a, given k. For values of $k = 0{\cdot}050(1)\ldots1{\cdot}000$ the value of a is given to seven decimal places. For $k<0{\cdot}050$ it is enough to assume $a = 1/k$.

Two statistical contexts in which the solution of this equation intervenes are discussed. The first is the equation for the maximum likelihood estimator of the Poisson parameter, λ, when the observed distribution is truncated and the zero group is missing. The second is the procedure suggested by Laplace for approximating to the difference quotients of zero. Laplace's transformation is compared with those put forward by De Moivre (1756), Binet& Szekeres [*Ann. Math. Statist.* (1957) **28**, 494-498], Arfwedson [*Skand. Aktuartidskr.* (1951) **34**, 121-132] and Hsu [*Ann. Math. Statist.* (1948) **19**, 273-277]. It is shown that Laplace's approximation is very good indeed and superior over the whole range to those recently proposed.

(Florence N. David)

2/226

CHIASSINO, G. (University of Bari, Italy) **0.2** (2.2)

On the analytic representation of survival curves—*In Italian*

Riv. Ital. Econ. Demogr. Statist. (1959) **13**, 657-660 (6 references, 1 table)

The author reconsiders a form first suggested by Perrone for the smoothing of the survival curve in a life table by which the validity of the analytic graduation is extended to ranges larger than those usually obtaining with other functions.

The suggested solution, according to Perrone, is based on a convenient transformation of the age variable: the modification introduced by the author is shown to produce better results with regard to the experience of the recent 1950-53 Italian table calculated for the male sector of the population.

(V. Levis)

2/227

HARTLEY, H. O. & LOFTSGARD, L. D. (Iowa State Univ. and North Dakota State Coll.) **0.8** (0.7)

Linear programming with variable restraints—*In English*

Iowa State Coll. J. Sci. (1958) **33**, 161-172 (2 references, 1 table)

Linear programming methods are becoming increasingly important for analysing economic situations. In particular, programming methods have been extensively applied to farm management problems, where maximum revenue defines optimum solutions. In these problems, the magnitude of attainable revenue depends on the alternative uses of resources and boundaries of the planning situation as represented by available resource supplies. This paper deals with the latter stipulation concerning resource supplies.

A more comprehensive method of programming is one that allows variation of resource supplies. That is, a method that determines continuous optimum solutions when one resource supply is varied within a relevant range and other resource supplies are held constant. One such method developed by Candler is a modified simplex solution for linear programming with variable capital restrictions.

This paper discusses a different modification of the simplex solution with continuous variation of one restraint. It is believed that the method proposed here has considerable advantages for programming on high-speed computers. The procedure is numerically illustrated by use of a sample problem from Candler's article.

(H. O. Hartley)

2/228

MIHOC, G. (University " C. I. Parhon ", Bucharest) **0.8** (2.3)

Some remarks in connection with the application of linear programming—*In Rumanian*

Revista de Statistică (1959) **12**, 13-17

In this paper the author suggests a method for solving the problem of linear programming under the hypothesis that the coefficients of the linear form to be extremised are normally distributed random variables with given means and variances.

The result obtained shows the statistical character of the linear programming method when applied to practical problems.

(M. Iosifescu)

2/229

NORDBOTTEN, S. (Statistisk Sentralbyrå, Oslo, Norway) **0.8** (11.5)

Linear programming and automatic computers—*In English*

Skand. Aktuartidskr. (1959) **42**, 61-72 (9 references, 1 chart)

Two applications of linear programming methods are indicated. The first problem treated is to find a plan of production which maximises the excess of exports over imports. The model employed is of the input-output type. The second problem concerns the allocation of sampling units in sampling a population of manufacturing establishments. The population is subdivided into industry groups. A second classification is into districts. Estimates are needed for the total production of each industry group and the total production within each district. It is assumed that the cost of sampling n_{ij} establishments in the ith group within the jth district is of the form a_{ij}-c_{ij}/n_{ij}. The minimisation of the cost with restraints placed on the allowable variances of the estimated totals then becomes a case of linear programming.

The simplex procedure for solving a linear programming problem is reviewed. A detailed flow-chart is presented for applying the procedure on an automatic computer.

(B. Matérn)

2/230

SHEPHARD, R. W. (University of California, Berkeley) **0.8** (–.–)

Non-linear programming. I. Linear representation of separable concave functions in production planning—*In German*

Qualitätskontr. Operat. Res. (1960) **5**, 131-136 (3 tables, 3 figures)

The method of non-linear programming as treated in this paper is a specific form of concave methods: it is supposed that the objective function (the function to be maximised or to be minimised) consists of a sum of separable concave functions of only one variable each, and that the constraints of the problem are linear. It is shown that problems of this type can be reduced to linear programmes.

The following example illustrates the above-mentioned method. A firm manufactures and sells three different commodities produced by only one machine with a limited capacity. This results in a linear inequality. The profit per unit of each commodity, being constant in certain intervals of production, decreases from interval to interval (price differentiation). The number of intervals is 2 for commodity A and 3 for B and C. After exceeding a fixed limit of production the profit becomes zero, consequently the profit function of each commodity is a concave function composed of sections linear within each interval, the sum of all commodity profit functions gives the objective function (total profit function). In order to reduce this programme to a linear form, for each possible unit profit of each commodity, non-negative variables must be introduced being permitted to become at most as large as the corresponding production interval. By this means the number of variables is increased, but the programme has become linear; as only one capacity constraint is assumed, it is solvable in an elementary manner without using the simplex method. The author points out that the approximation to a concave function with continuously decreasing derivation by means of a polygon leads to similar programmes.

Hereafter, the above example is extended by increasing the number of machines producing the three commodities. The capacity of the four new machines is restricted and so there are four linear inequalities as capacity restrictions. A matrix (a_{ij}) indicates the

continued

SHEPHARD, R. W. (University of California, Berkeley) **0.8** (–.–) *continued*

Non-linear programming. I. Linear representation of separable concave functions in production planning—*In German*

Qualitätskontr. Operat. Res. (1960) **5**, 131-136 (3 tables, 3 figures)

number of hours used on the ith machine to produce one unit of the jth commodity. The procedure of making the programme linear is the same as that cited above.

Finally, the author presents some ideas on the applications in practice, showing how to remove bottle-necks by marginal interpretation. He finishes by calling attention to the primary-dual relation and illustrates the notion of the shadow prices (opportunity costs) to demonstrate how to determine the marginal productivity by means of the duality theorem.

(W. Dinkelbach)

SHEPHARD, R. W. (University of California, Berkeley) **0.8** (-.-)

Non-linear programming. II. The Kuhn-Tucker theorem, quadratic programming and the gradient method—*In German*

Qualitätskontr. Operat. Res. (1960) **5**, 144-149 (4 references)

In part one of this paper [*Qualitätskontr. Operat. Res.* (1960) **5**, 131-136: abstracted in this present journal No. **2**/231, **0.8**] the author is concerned with the linear representation of separable concave functions in the field of production planning.

In part two of this paper the Kuhn-Tucker theorem is stated and discussed; this includes its application to quadratic programming (the algorithm of Wolfe). This is followed by an outline of the gradient method. This paper contains a bibliography of the original papers on those subjects.

(W. Dinkelbach)

2/233

STAHLKNECHT, P. (Karl-Marx-Universität, Leipzig) **0.8** (-.-)

Linear programming applied to problems of managerial science with changing conditions—*In German*

Wissensch. Zeit. Karl-Marx-Univ. Leipzig (1958/59) **8**, 693-702

The present article is part of the author's thesis at the Faculty of Science, Karl-Marx-University of Leipzig. The author briefly discusses the problems underlying linear programming. In the following paper he quotes the methods most frequently used for finding optimal solutions of linear programmes and describes the particular aspects to which he devotes this paper.

In management science, linear programmes often occur for which the bounds of some variables are not constant. One is not therefore mainly interested in the solution of one fixed linear programme in the usual sense of the word, but particularly solutions which show how the result is influenced by changes of the bounds.

In developing a method of constructing such solutions the author starts with a special problem; this is treated numerically *in extenso*. The experiences gained hereby can be generalised without difficulties, leading to a certain procedure applicable in general cases. It consists mainly in the repeated performance of some simple mathematical operations. The following problems may be treated in the manner described by the author:

$$p = p_o + \sum_{i=1}^{n} p_i x_i = \text{Min}$$

under the conditions

$$x_k = v_{k_o} + \sum_{i=1}^{n} v_{k_i} x_i \geqq 0 \quad k = n+1, \ldots, m$$

$$0 \leqq x_i \leqq r_i \qquad i = 1, \ldots, n$$

The quantities p_o, p_i, v_{k_o}, v_{k_i} ($i = 1, \ldots, n$; $k = n+1, \ldots, m$) are given constants, while information concerning the bounds r_i is not available in advance.

As an illustration, the well-known blending problem is used, in the case where the optimal composition of a mixture of n gases is to be found, and whose calorific value has to remain in given bounds while the proportion of sulphur in the mixture must be below a certain limit. In addition it is required to produce a certain quantity of the mixture. The solution especially has to show the influence of certain sorts of gases on the optimum composition.

In a final section practical experiences are reported. Only the hand computation is treated; the use of the electronic computers is not considered.

(W. Knödel)

2/234

AUSTIN, T. L., FAGEN, R. E., PENNEY, W. F., & RIORDAN, J. **1.3** (11.3)

(Silver Spring, Md.; Hughes Aircraft Co.; Silver Spring, Md.; and Bell Telephone Labs.)

The number of components in random linear graphs—*In English*

Ann. Math. Statist. (1959) **30**, 747-754 (3 references, 1 table)

Given n distinct (labelled) points, a random linear graph is given by m selections of pairs of points made independently and at random, each of the $\binom{n}{2}$ pairs having the same chance of selection at each trial, upon joining each selected pair by a (labelled) line. A " component ", or " piece ", of such a graph is a collection of k of the n points, such that any two of the k points are connected by some path (possibly through other of the k points), and such that no point other than these k points is connected to any of the k points.

Let T_{nmp} denote the number of such graphs having n points, m lines, and p parts; let $C_{nm} = T_{nm1}$ denote the number of connected graphs having n points and m lines. Let

$$T(x, y, z) = \Sigma\Sigma\Sigma\, T_{nmp} \frac{x^n}{n!} \frac{y^m}{m!} z^p, \text{ and}$$

$$C(x, y) = \Sigma\Sigma\, C_{nm} \frac{x^n}{n!} \frac{y^m}{m!}.$$

The authors show that $T(x, y, z) = \exp[zC(x, y)]$, and exhibit specific formulae for T_{nmp} and C_{nm} resulting from the equality just given. The results described are unwieldy for large n. Consequently, the authors determine by other means formulae for $T_{n, n-1, 1} = C_{n, n-1}$ and for $T_{n, n, 1} = C_{n, n}$, and quote the explicit forms for $T_{n, m, n-j}$, $j = 1, 2, 3$.

The average number of " components " is written down in explicit but elaborate form. An alternative development is presented allowing certain asymptotic approximations to be used. Finally, the approximation is shown, compared with actual values, for M_{nn}, $n = 3$, 4, ..., 10.

Note: In expression (13), the factor $\binom{m}{j}$ is missing inside the summation. To obtain expression (15) from (14), multiplication should be by $x^n/n!$, not $x^{n+1}/(n+1)!$.

(J. A. Lechner)

2/235

BARTON, D. E. & DAVID, Florence N. (University College, London) **1.3** (4.7)

A collector's problem—*In English*

Trab. Estadíst. (1959) **10**, 75-88 (3 references, 2 tables)

This paper deals with two aspects of the classical balls-in-boxes occupancy problem; first the case when the number of balls to be dropped is fixed and the number of empty boxes is a random variable, and second the case in which the number of empty boxes is fixed and the number of balls to be dropped is the variable.

N identical compartments are supposed and one further control box; n balls in all are dropped independently. The probability that $n-j$ balls will fall in the control box and j in the N compartments to give k empty compartments and $N-k$ occupied is found: the probability is also calculated of k empty cells and at least $n-b$ balls in the control box. The factorial moments of k, conditional on $n-j$ balls in the control box, and the conditional factorial moments of k when there must be at least $n-b$ balls in the control box is found.

The obverse of the problem, when n becomes the random variable, is considered under the sequential aspect; it is supposed that balls are dropped one by one independently and at random, and that this process continues until there are at least $(N-k)$ occupied boxes and b in the control box: or, there are precisely $N-k$ occupied boxes and at least b in the control box. The probability density function, the moments of n and some asymptotic distributions are studied.

Finally, the exact evaluation of some upper percentage points are given and the error involved when these points are calculated by means of the asymptotic distributions previously mentioned is stated.

(J. M. Garcia)

BOROVKOFF, A. A. (Moscow) **1.5** (10.0)

Limit theorems on the distributions of maximum of sums of bounded latticed random variables. I.—*In Russian*

Teor. Veroyat. Primen. (1960) **5**, 137-171 (22 references)

The bounded latticed identically distributed random variables $\xi_1, \xi_2, \ldots,$ are considered by the author in this paper. The local and integral theorems for the first passage time η_x over the barrier $x>0$ in the random wanderings along the straight line with the quantity of jump ξ_k are studied. The formulas for $\Pr(\eta_x = n)$ and $\Pr(\eta_x > n)$ in obvious form are obtained for the full "spectrum" of values x, beginning with $x = o(n)$ until x, is equivalent to the product maximum jump ξ_k by n.

The theorems for $\Pr(\eta_x > n)$ simultaneously are integral theorems for the maximum of sums $\sum_{k \leq v} \xi_k (v = 1, 2, \ldots, n)$. The formulas for first moments η_x and the distribution of the quantity of the first excess over the barrier x are also obtained.

(A. A. Borovkoff)

2/237

ERDÖS, P. & RÉNYI, A. (Math. Inst., Hungarian Academy of Sciences, Budapest) **1.3** (1.5)

On random graphs. I.—*In English*

Publ. Math., Debrecen (1959) **6**, 290-297 (8 references)

Let us consider all possible graphs with n given vertices $P_1, P_2, \ldots, P_n$ and N edges. Let us select one of them at random so that each of these graphs should have the same chance to be selected. Such a random graph should be denoted by $\Gamma_{n,N}$. Put $N_c(n) = [\frac{1}{2}n \log n + cn]$. Denoting by $P_k[n, N_c(n)]$ $(k = 0, 1, \ldots)$ the probability that the greatest connected component of $\Gamma_n, N_c(n)$ consists of $n-k$ points, the authors show that for $n \to +\infty$ this distribution tends to Poisson's law with mean value $\exp(-2c)$. For $k = 0$ this theorem implies that the probability that $\Gamma_n, N_c(n)$ should be connected tends for $n \to +\infty$ to $\exp(-e^{-2c})$. This includes previous (unpublished) result of Erdös and Whitney. It is shown further that in the above case when $\Gamma_n, N_c(n)$ consists of a connected component having $n-k$ points, then the remaining k points are all isolated with probability tending to 1. The authors further determine the limiting distribution of the random variable Vn defined in the following way: the edges of a graph with n possible vertices are chosen successively so that after each step the next edge is chosen randomly from the possible edges not yet chosen, until the graph becomes completely connected. Vn denotes the number of edges of the resulting graph.

(I. Vincze)

2/238

HALDANE, J. B. S. (Indian Statist. Inst., Calcutta) **1.0** (2.1)

The addition of random vectors—*In English*

Sankhyā (1960) **22**, 213-220 (3 references)

The author considers the addition of random vectors $\boldsymbol{x}$, such that $\boldsymbol{x}$ has uniform distribution over spheres $\| x \| = c$ for each c. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are two such independent vectors then it can be seen that $z = \boldsymbol{x}+\boldsymbol{y}$ is also uniformly distributed over spheres. The first five cumulants of $\| z \|$ are derived in terms of the cumulants of $\| \boldsymbol{x} \|$ and $\| \boldsymbol{y} \|$. The author defines what are called vectorial cumulants θ_r of $\| \boldsymbol{x} \|$ which are functions of the cumulants κ_r of $\| \boldsymbol{x} \|$, with the following properties:

(i) θ_r is a function of $\kappa_1, \ldots, \kappa_n$ only;

(ii) κ_n is a function of $\theta_1, \ldots, \theta_r$ only;

(iii) $\theta_r(\| \boldsymbol{x}+\boldsymbol{y} \|) = \theta_r(\| \boldsymbol{x} \|)+\theta_r(\| \boldsymbol{y} \|)$.

Explicit expressions for θ_r in terms of κ_n and vice-versa have been given. The vectorial cumulants θ_r are useful in studying the distribution of the resultant sum of any number of random vectors $\boldsymbol{x}$, $\boldsymbol{y}$, z, ... First, θ_r of the resultant is obtained by using the addivity property and from them the corresponding κ_n are computed.

It has been noted for the gamma distribution that $\theta_r = 0$ for $r \geqq 2$. The author mentions possible applications of the distribution of the resultant sum of random vectors in some problems in Physics and the study of migration in Demography.

(S. R. S. Varadhan)

2/239

HODGES, J. L., Jr. & LE CAM, L. (Univ. of California, Berkeley) **1.5** (2.5)

The Poisson approximation to the Poisson binomial distribution—*In English*

Ann. Math. Statist. (1960) **31**, 737-740 (5 references)

Let X_i be a random variable which can be realised as either zero or one, and let $\Pr\{X_i = 1\} = p_i$, $i = 1, 2, \ldots, n$. The random variable $S = \sum_1^n X_i$ is sometimes said to have a distribution called " Poisson binomial ". It is known that S tends to become a Poisson variate in the limit (on n) when $\Sigma p_i = \lambda$ is fixed and $\alpha = \max\{p_i\}$ tends to zero. It can also be shown that the absolute difference between the distribution of S and that of a Poisson variate with parameter Σp_i is bounded by $C\alpha^{1/5}$, where C is independent of n and the p_i. This paper presents a simple proof that the bound $C\alpha^{1/5}$ just stated may be improved to $3\alpha^{1/3}$.

The proof is accomplished in three stages: the authors are concerned with the pedagogical as well as the mathematical aspects of their work. In this connection, their theorems are stated as approximation theorems rather than limit theorems, with the corresponding limit theorems indicated briefly at the conclusion.

(D. H. Shaffer)

2/240

MANDL, P. (Math. Inst., Czechoslovak Academy of Sciences, Prague) **1.5** (10.1)

On limit properties of probabilities in groups of states of an homogeneous Markoff chain—*In Russian*

Čas. pěst. math. (1959) **84**, 140-149 (3 references)

Let us consider a finite homogeneous Markoff chain. If T is a class of states of this chain and $w \in T$, then the symbol $\Pr(x_n = w \mid T^n)$ denotes the probability that the chain is in state w at the moment n provided that it has been in states of the set T at all moments between 0 and n inclusive. The author studies limit properties of these probabilities for $n \to \infty$.

The class T is called regular if the matrix formed by probabilities of mutual transitions between states of T is non-decomposable and aperiodic. It is proved in the paper that if T is regular, the corresponding limits $\lim_{n\to\infty} \Pr(x_n = w \mid T^n)$ always exist (for every $w \in T$), are positive and do not depend on the initial probability distribution.

Further, the author investigates the case when the class is divided into sub-classes $T_1, \ldots, T_l$ such that, if for any two states $w_1, w_2 \in T$ the transition probabilities from w_1 to w_2 and from w_2 to w_1 are both positive: then, w_1 and w_2 belong to the same sub-class. The system of these sub-classes is partially ordered by the relation $T_j < T_k$ if the transition from T_j to T_k has positive probability; the corresponding transition path being composed of states belonging to T only. Let us denote by ρ_j the characteristic number of the matrix corresponding to T_j, with maximum absolute value. Let $\rho = \max_{1 \leq j \leq l} \rho_j$, and let k be the greatest number such that there exist k sub-classes $T_{j_1} < \ldots < T_{j_k}$ with characteristic numbers $\rho_{j_i} = \rho$.

If all classes T_j are regular and the chain begins its evolution in a state belonging to a class which is minimal with respect to the partial ordering $<$, then for every $w \in T$ there exists the limit $\lim_{n\to\infty} \Pr(x_n = w \mid T^n)$. It is positive if, and only if, w belongs to a sub-class T_j such that there exist k sub-classes $T_{j_1} < T_{j_2} < \ldots < T_{j_k} \leqq T_j$ with $\rho_{j_i} = \rho (i = 1, 2, \ldots, k)$. If there exists only one

continued

MANDL, P. (Math. Inst., Czechoslovak Academy of Sciences, Prague) **1.5** (10.1) *continued*

On limit properties of probabilities in groups of states of an homogeneous Markoff chain—*In Russian*

Čas. pěst. math. (1959) **84**, 140-149 (3 references)

such sequence $T_{j_1} < T_{j_2} < \ldots < T_{j_k}$ with $\rho_{j_i} = \rho (i = 1, 2, \ldots, k)$ then the limits do not depend on the initial probability distribution provided that the chain starts in a minimal class.

In these cases the probability $\Pr(T^n)$ that the chain remains in states of T at any moment between 0 and n is of such an order that

$$\lim_{n\to\infty} [\Pr(T^n)/n^{k-1}\rho^n] > 0.$$

(F. Zítek)

MASLOFF, K. A. (Kharkoff University) **1.5** (-.-)

On certain limit theorems in probability theory—*In Russian*

Teor. Veroyat. Primen (1960) **5**, 54-83 (10 references)

Let $\xi_{n,k}(1 \leqq k \leqq k_n;\ n = 1, 2, ...)$ be a sequence of a series of independent random variables, $\phi(x, y)$ is any function and the random variables $\zeta_{n,k}$ are determined as

$\zeta_{n,1} = \xi_{n,1}, \zeta_{n,k+1} = \phi(\zeta_{n,k};\ \xi_{n,k+1}), k = 1, 2, ..., k_n - 1.$

The author finds a sufficient condition for the existence of a limit distribution $\zeta_{n,k_n}(n \to \infty)$ and for the form of this distribution.

(K. A. Masloff)

2/243

NARAYANA, T. V. (Institut H. Poincaré and McGill Univ., Quebec) **1.3** (1.4)

A partial order and its applications to probability theory—*In English*

Sankhyā (1959) **21**, 91-98 (4 references)

An r-partition of n is a set of integers $(t_1, ..., t_r)$ with $t_1 + ... + t_r = n$. The partial order in question is the following: an r-partition $(t_1, ..., t_r)$ dominates $(t'_1, ..., t'_r)$ if $t_1 + ... + t_j \geqq t'_1 + ... + t'_j (j = 1, 2, ..., r-1)$. The author shows, using a geometric argument, that the total number of " dominations " among r-partitions is

$\binom{n}{r}\binom{n}{r-1}/n.$

One application of this partial order is the following problem in probability theory: " it is known that two candidates A, B for election will secure m and n votes $(m > n)$ respectively, what is the probability that throughout the scrutiny A holds a L-lead over $B(1 \leqq L \leqq m-n)$? " The author also discusses the connection with the following game: there are two coins 1 and 2 with p_1, p_2 as the probability of heads occurring when the coins are tossed. The first toss is made with coin 1 and each succeeding trial is made with coin 1 or 2 according to whether the result of the preceding toss was tails or heads.

(R. Ranga Rao)

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PROKHOVNIK, S. J. (University of New South Wales) **1.3** (–.–)

Pattern variants on a square field—*In English*

Psychometrika (1959) **24**, 329-341

A quantitative approach to the psychology of pattern recognition requires knowledge of the number of possible variants of any particular pattern. The general solution for the number (p/m^2) of pattern variants that p counters can form on a square network of m^2 positions obtained by elementary group theory. The exact solution is given in terms of the different types (symmetric, asymmetric, etc.) of patterns possible in an approximate formula for the total number of patterns is also developed

Different considerations apply for odd and even cases, both for networks and for number of points forming the pattern. Further, symmetrical patterns of p points over an m^2 network are possible only if p is a multiple of four and, in the case when m is odd, also if $p-1$ is a multiple of four.

A table is given showing a selection of numerical evaluations of G_8, G_4, G_2, and G_1, and the total number of patterns for networks up to 6^2 positions. It is clear from this table that the total number of symmetrical patterns is only a small ratio of all patterns possible for a particular case, and further, that this ratio decreases with increasing m, according to a square root law. This small ratio leads to two generalisations of practical consequences. First, in any physical problem involving the elucidation of the pattern of the structure, it would be clearly advantageous to seek any evidence of symmetry within the structure. If symmetry is displayed, not only is the number of patterns possible reduced significantly but also these can be listed and checked systematically under the groups and sub-groups of G_8, G_4, and G_2. Secondly, for p, $m>2$ greater than the total number of patterns (p/m^2) is of the order of $1/8 \binom{m^2}{p}$ though always greater than this number.

An appendix giving proofs of the theorems presented is included.

(R. E. Stoltz)

2/245

RANGA RAO, R. & VARADARAJAN, V. S. (Indian Statist. Inst., Calcutta) **1.5** (–.–)

A limit theorem for densities—*In English*

Sankhyā (1960) **22**, 261-266 (8 references)

In this paper the problem of almost everywhere convergence of the density of the normalised sum of a sequence of independent and identically distributed random variables to the density of the standard normal distribution is studied. In particular the following results are proved.

Let ξ_1, ξ_2, ... be a sequence of independently and identically distributed random variables with mean zero and variance unity; $F_n(x)$ the distribution function of the normalised sum $x_n = (\xi_1+\ldots+\xi_n)/\sqrt{n}$, and $p_n(x)$ the density of the absolutely continuous part of $F_n(x)$. If for at least one n, $F_n(x)$ is non-singular then $p_n(x)$ converges almost everywhere to $\phi(x) = (2\pi)^{-\frac{1}{2}}\exp(-\frac{1}{2}x^2)$. This result is valid even if ξ_1, ξ_2, ... are random vectors with mean zero and dispersion matrix Σ, provided $\phi(x)$ is replaced by the corresponding normal density with mean zero and dispersion matrix Σ.

It is pointed out that the above result can be extended to the case of convergence to stable laws. Generalising a result of Gnedenko, the following estimate is obtained: there exist positive constants c_1, c_2, $\lambda(\geqq 0, <1)$ such that $L\{x : | p_n(x)-(x) | \leqq c_1/\sqrt{n}\} \leqq c_2\lambda^n$, L being Lebesgue measure.

As an application of the above-mentioned result, the authors prove the following theorem in the case of random vectors taking values in k-dimensional Euclidean space E_k:

let the common distribution function $F(x)$ of ξ_1, ξ_2, ... be absolutely continuous, $\Lambda_1(x)$, $\Lambda_2(x)$ be two fixed independent linear functions on E_k, $p_n(x \mid y)$ the density of the conditional distribution of $\Lambda_1(\xi_n)$ given $\Lambda_2(\xi_n)$ and $\phi(x \mid y)$ the corresponding conditional density of the normal distribution. Then $\lim p_n(x \mid y) = \phi(x \mid y)$ almost everywhere in x for almost all (Lebesgue) y. In particular,

$$\lim_{n\to\infty} \sup_{A} | P_n(A \mid y) - P(A \mid y) 1 = 0$$

for almost all y, where the supremum is taken over all Borel sets, and $P_n(A \mid y)$, $P(A \mid y)$ represents the corresponding conditional distributions.

(K. R. Parthasarathy)

2/246

ROZANOFF, Y. A. (Moscow) **1.5** (-.-)

A central limit theorem for additive random functions—*In Russian*

Teor. Veroyat. Primen. (1960) **5**, 243-246 (4 references)

The additive random functions $H(\Delta)$ of semi-interval $\Delta = [s, t)$, satisfying the strong mixing condition: $A \in \mathfrak{M}^{t}_{-\infty}$, $B \in \mathfrak{M}^{\infty}_{t+\tau}$

$$\alpha(\tau) = \sup_{t} \sup_{A,\, B} \left| \Pr(AB) - \Pr(A)\Pr(B) \right| \to 0 \quad (\tau \to \infty)$$

are considered. Let $\alpha(\tau) = O(\tau^{-1-\varepsilon})$, $\delta > 2/\varepsilon$

$$M \left| H(\Delta_0) - MH(\Delta_0) \right|^{2+\delta} \leqq M_0,$$

for all $\Delta_0 = [t_0, t+t_0)$,

$$0 < \underline{\lim}_{|\Delta| \to \infty} \frac{\mathscr{D}H(\Delta)}{\Delta} \leqq \overline{\lim}_{|\Delta| \to \infty} \frac{\mathscr{D}H(\Delta)}{\Delta} < \infty$$

then for $|\Delta| = t - s \to \infty$

$$\Pr \left\{ \frac{H(\Delta) - MH(\Delta)}{\sqrt{\mathscr{D}H(\Delta)}} < x \right\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}}\, du.$$

(Y. A. Rozanoff)

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2/248

ADKE, S. R. (University of Poona) **2.0** (-.-)

Generalised affinity and a class of distance functions—*In English*

Proc. Nat. Inst. Sci. India A (1959) **5**, 104-110 (7 references)

Bhattacharya has introduced the concept of affinity between probability laws, which is a measure of divergence or distance between two statistical populations [*Bull. Calcutta Math. Soc.* (1943) **35**, 99-109], providing a generalisation of Mahalanobis' distance between two multivariate normal distributions. Some properties of Bhattacharya's affinity and the distance functions associated with it have been studied by Adhikari & Joshi [*Publ. Inst. Statist. Paris* (1956) **5**, 57-74]. The present author generalises the definition of affinity and considers some properties of a class of distance functions associated with the generalised affinity.

(B. Raja Rao)

2/249

BLACKWELL, D. & HODGES, J. L., Jr. (University of California, Berkeley) **2.1** (1.1)

The probability in the extreme tail of a convolution—*In English*

Ann. Math. Statist. (1959) **30**, 1113-1120 (6 references, 2 tables, 1 figure)

For the case of independent and identically distributed lattice variables, the authors give approximations to $\Pr\{X_1+\ldots+X_n = na\}$ and $\Pr\{X_1+\ldots+X_n \geqq na\}$ where $\mathscr{E}X_1 < a < \sup X_1$. It is assumed that the moment generating function of X_1 exists in a neighbourhood of the origin. The derivation of the approximations involves defining, through the moment generating function, an auxiliary sequence of independent and identically distributed random variables $Y_1, \ldots, Y_n$, taking the same values as X_1, so that $\mathscr{E}Y_1 = a$. From the definition it follows that the "tail" probabilities above are related to "central" probabilities of the distribution of $Y_1+\ldots+Y_n$. The approximations are then provided by an inversion argument in terms of the characteristic function of Y_1 and involve the central moments of Y_1. Two approximations are given for each probability and for each case, the approximations have relative errors of order n^{-1} and n^{-2}. A conjecture is given concerning an analogous approximation in the non-lattice case.

(N. D. Ylvisaker)

2/250

DABONI, L. (University of Trieste, Italy) **2.5** (1.1)

A property of Poisson distributions—*In Italian*

Boll. Un. Mat. Ital. (1959) **14**, 318-320 (1 reference)

The author, with reference to the proof by Lamperti [*The American Mathematical Monthly* (1959)], gives a new proof of a theorem according to which if, in a series of Bernoulli trials their number N is random, then the numbers N_A of successes and $N_B = N - N_A$ of failures are stochastically independent only when N follows a Poisson distribution.

The argument is based on the fact that the generating functions of Poisson distributions are the unique solutions of a functional equation of the generating function of N based on the condition that N_A and N_B are stochastically independent.

(G. Chiassino)

2/251

EKMAN, G. (Inst. Math. Statist., Univ. Stockholm) **2.1** (8.7)

Approximate expressions for the conditional mean and variance over small intervals of a continuous distribution—*In English*

Ann. Math. Statist. (1959) **30**, 1131-1134 (2 references)

The paper presents approximations to the conditional mean and conditional variance of an absolutely continuous random variable, given that the random variable is restricted to a small interval of its range. For this purpose the density function f of the random variable is assumed to have continuous derivatives up to the fourth order. The approximations result from the combination of expansions for the first three " moments " of the random variable over the interval of restriction, in powers of the length of this interval, with the Taylor series for $\log f$, applied to the same interval. The work stems from the problem of determining optimum stratification points for stratified sampling with intent to estimate the mean of a continuous random variable. Application of the results is made in this paper to the determination of stratification points when strata sample sizes are to be allocated in proportion to the probabilities of the strata. The corresponding problem of selection of optimum stratification points to be used with a minimum variance linear estimate of the mean of a continuous random variable is discussed by the author in a previous paper [*Ann. Math. Statist.* (1959) **30**, 219-229; abstracted in this journal No. **1**/268, **8.1**].

(J. D. Esary)

2/252

GEFFROY, J. **2.6** (1.5)

A contribution to the theory of extreme values—*In French*

Publ. Inst. Statist. Paris (1959) **8**, 36-121, 124-185 (26 references)

The first chapter of this long and detailed paper is devoted to a study of the upper and lower asymptotic bounds of the extreme values of a sample. After an introduction including a section dealing with notation, the author investigates stochastic convergence, upper and lower probability bounds for extreme values, almost certain upper and lower bounds and almost complete upper and lower bounds. Twelve theorems are stated in this chapter which concludes with a note on an unusual class of functions.

The study of stability in probability of extreme sample values forms the subject of the next chapter. The author, after stating that this study follows logically from the results of his first chapter, proceeds to discuss stability of random variables, inversion of a frequency function and the stability of the sum of random variables, and a theorem by Gnedenko [*Ann. Math.* (1943) **44**, 423] and notes that this theorem furnishes a stability condition. The latter part of the chapter deals with the study of linked extremes of a sample from the point of view of stability and ends with the equivalence in probability of the extreme values of a sample. This chapter contains many theorems and their corollaries.

Chapter three dealing with almost complete stability of the extreme values of a sample is followed in chapter four by a study of almost sure stability.

In chapter five the author gives some consideration to the idea of limiting independence of two random variables and its application to the study of the stability of the " middle " of the sample and range. In the following chapter a detailed study is made of certain limit laws and the paper concludes with an investigation of the properties of two-dimensional samples.

In the two parts of this paper there are nearly eighty theorems and their corollaries together with an extensive bibliography.

(Mlle. Gervaise)

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GHURYE, S. G. & WALLACE, D. L. (University of Chicago) **2.1** (2.9)

A convolutive class of monotone likelihood ratio families—*In English*

Ann. Math. Statist. (1959) **30**, 1158-1164 (5 references)

If $f(\cdot, \theta)$ and $g(\cdot, \theta)$ define two one-parameter families of densities on the real line or the integers, which have the monotone likelihood-ratio property, the family defined by convolution $(f*g)$ will not generally have the property. A sufficient condition for the convolution to have the monotone likelihood-ratio property is that for every θ, the one-parameter families with parameter ξ defined by $h(x, \xi) = f(x-\xi, \theta)$ and $k(x, \xi) = g(x-\xi, \theta)$ also have the property. The condition is shown not to be necessary.

A corollary and application is: let Y be a real random variable and, conditionally on Y, let $X_1, \ldots, X_n$ be independent Bernoulli random variables. Assume that for each i, $\Pr\{x_i = 1 \mid Y\}$ is a nondecreasing function of Y. Then, with Y as a parameter, the family of distributions of X_i has the monotone likelihood-ratio property for each i, and so does that of $S = X_1 + \ldots + X_n$. Whatever the distribution of Y, the conditional distribution function of Y given $S = a$ lies to the left of that for $S = b$ if $b > a$.

Some generalisations of the results to very special multivariate distributions are given.

(D. L. Wallace)

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GUMBEL, E. J. (Columbia University, New York) **2.0** (2.1)

Multivariate distributions with given margins and analytical examples—*In English*

Bull. Int. Statist. Inst. (1960) **37**, III 363-373 (6 references)

The aim of this paper is to derive a general and simple system which will connect any two given distributions $F(x)$ and $G(y)$ to a bivariate distribution $H(x, y)$, such that the given distributions are marginal. This system is obtained by a generalisation of the construction due to Morgenstern [*Mitt. Math. Statist. Würzburg* (1956) **8**, 234-235]. The result, which is also generalised to any finite number of marginal distribution, is

$$H(x, y) = F(x)G(y)1+\alpha[1-F(x)][1-G(y)],$$

where α is any real number with $-1 \leqq \alpha \leqq 1$. There is an infinity of bivariate distributions, such that the two given ones are marginal, which is certainly not exhausted by the given infinite system. Fréchet has shown, for example, that $H_1(x, y) = \max[F(x)+G(y)-1, 0]$ and $H_2(x, y) = \min[F(x), G(y)]$ are bivariate distributions for which, for all x and y, $H_1(x, y) \leqq H(x, y) \leqq H_2(x, y)$ for any $H(x, y)$ having $F(x)$ and $G(y)$ as marginal distributions.

Included in the system is a bivariate distribution with uniform marginal distributions, for which the regression curves are linear, and a bivariate distribution with normal marginal distributions where the regression curves are not linear. In the system the coefficient of correlation varies only from $-1/3$ to $1/3$: the limits are attained for uniform marginal distributions. Hence a correlation coefficient 0·30 means strong correlation for a bivariate distribution belonging to this system.

Some further interesting examples of the unexpected behaviour of distributions are given.

(J. T. Runnenburg)

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GURLAND, J. (Iowa State University, Ames) **2.5** (4.0)

Some applications of the negative binomial and other contagious distributions—*In English*

Amer. J. Publ. Health (1959) **49**, 1388-1399 (39 references)

Much of the experimentation in the biological and medical sciences pertains to statistical distributions which are far from normal, for example the negative binomial and other contagious frequency distributions.

The binomial and Poisson distributions are described, with an example, and their relation to the negative binomial discussed. If the Poisson distribution is compounded with a Gamma distribution, that is if the Poisson parameter λ itself follows a Gamma distribution, the resulting distribution is negative binomial; this can be referred to as " apparent contagion ". A model based on true contagion also yields the negative binomial as a limiting case, and so it is apparently impossible from the data to distinguish between the two models.

Another mathematical model that leads to the negative binomial distribution arises when the Poisson distribution is generalised through a logarithmic distribution; for example, if the number of bacterial colonies in a field follows a Poisson distribution and the number of bacteria per colony a logarithmic distribution, then the distribution of bacteria in each field is a negative binomial.

The method of moments and the method of frequencies are described for estimation and fitting. Other compound and generalised distributions are discussed, and, along with the negative binomial, fitted to some medical data.

One example concerns the distribution of dental caries in twelve-year-old children. A Pascal distribution compounded with a Poisson distribution (which is of the same form as a Poisson distribution generalised through a Pascal distribution) is fitted by a simple combination of the method of moments and method of frequencies; the fit is about as close as that accomplished by the negative binomial distribution when the method of maximum likelihood is used. Another example concerns the sickness distribution of shunters over a period of time. Here the negative binomial gives a poor fit even when determined by maximum likelihood. Using the method of moments the best fit is obtained with a Neyman Type A distribution (which arises as a Poisson distribution either compounded with or generalised through another Poisson distribution).

(R. C. Elston)

2/256

KHAMIS, S. H. (Amer. Univ., Beirut and Food & Agric. Org., Rome) **2.7** (0.3)

Incomplete gamma functions expansions of statistical distribution functions—*In English*

Bull. Int. Statist. Inst. (1960) **37**, III 385-396 (14 references). Discussion abstracted No. **2**/258

Incomplete gamma function series expansions of density and cumulative distribution functions are developed by the author in this paper. The coefficients of the successive terms depend upon the moments of the unknown distribution function. Finite difference methods are developed for the computation of the coefficients which result in very simple computational schemes. The inversion of the resulting series is also discussed.

In particular if $\theta_n(x) = [\alpha^n/\Gamma(n)]x^{n-1} . e^{-\alpha x}$, for positive n, x and α, and if $\Phi_n(x) = \int_0^x \theta_n(x)dx$, the two equivalent series expansions for a density function $f(x)$ given by

$$\sum_{r=0}^{m} A_r \Delta^r \theta_n(x) \text{ and } (-\alpha) \sum_{r=0}^{m} B_r \Delta \Phi_{n+r-1}(x),$$

and for the corresponding cumulative distribution function given by

$$\sum_{r=0}^{m} A_r \Delta^r \Phi_n(x) \text{ and } \sum_{r=0}^{m} B_r \Phi_{n+r}(x)$$

are obtained: where the A_r are easily computed from the leading differences of the ratios of the moments of the unknown distribution to the corresponding moments of $\theta_n(x)$ and the B_r from the A_r.

In the above, the difference operators Δ^r apply to the subscripts of θ and Φ. The finite difference properties are used in conjunction with the inverse Taylor Theorem for inverting the above expansions.

The following are two by-products of the methods used in the paper:

(i) The Laguerre polynomial $L_r(x)$ defined by

$$(-d/dx)^r[x^r\theta_n(x)] \equiv L_r(x)\theta_n(x)$$

is shown to be related to the differences of $\theta_n(x)$ and $\Phi_n(x)$ by the identities

$$L_r(x) \equiv \frac{\Gamma(n+r)}{\Gamma(n)} \cdot \frac{\Delta^{r+1}\Phi_{n-1}(x)}{\Delta\Phi_{n-1}(x)} \equiv \frac{\Gamma(n+r)}{\Gamma(n)} \cdot \frac{\Delta^r\theta_n(x)}{\theta_n(x)}$$

(ii) The finite difference properties of $\theta_n(x)$ and $\Phi_n(x)$ are useful in the tabulation of these two functions. The method developed by the author for this purpose has been used by Hartley & Pearson, E. S. in computing a new χ^2 table [*Biometrika* (1950) **37**, 313-325].

The paper contains a number of illustrations of the application of the expansion and inversion techniques developed therein.

(S. H. Khamis)

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KHAMIS, S. H. (Report of discussion on a paper by) **2.7** (0.3)

Incomplete gamma functions expansions of statistical distribution functions—*In English*

Bull. Int. Statist. Inst. (1960) **37**, I 102-103

Those taking part were Hotelling, Neyman and Martin.

Hotelling observed that this kind of expansion was particularly suitable for distributions of positive definite quadratic forms. This series of expansions offered possible help in studying effects of moderate deviations from standard assumptions when standard methods were applied.

Neyman said that the method of deducing these formulae could be reduced to the minimisation of a certain sum of squares with convincingly devised weights. This method suffered from the disadvantage that on frequent occasions it led to approximation curves having negative ordinates.

The author replied that he agreed with Neyman as to the difficulties regarding negative ordinates; this occurred in all series expansions. The only method he knows which, once successfully applied, would lead to a non-decreasing probability function was the method developed by Hartley & Khamis [*Biometrika* (1947) **34**, 340-351]. This method had an arbitrary parameter and the resulting approximation was satisfactory. However, in respect of series expansions no solution had yet been found.

(W. R. Buckland)

2/258

KHATRI, C. G. & SHAH, B. K. (University of Baroda, India) **2.1** (0.2)

Method of fitting certain curves of force of mortality over the whole range—*In English*

J. Univ. Baroda (1959) **8**, 33-40 (3 references)

Two new curves which will adequately fit data on force mortality μ_x are discussed. They are

(i) $y_x = A + B_1 d_1^x + B_2 d_2^x$;

(ii) $y_x = A + Hx + B_1 d_1^x + B_2 d_2^x$.

The method of fitting known as internal least-squares, introduced by Hartley [*Biometrika* (1948) **35**, 32-45] is adopted. It is assumed that the values of y_x are given for values of x in an arithmetic progression. For both curves, separate equations for estimation of the constants are obtained for the three cases:

(i) x taking values 0, 1, 2, ..., n;

(ii) x taking values $-m$, $-(m-1)$, ..., 1, 0, 1, 2, ..., m; and

(iii) x taking values $-m+\frac{1}{2}$, $-(m-1)+\frac{1}{2}$..., $-\frac{1}{2}$, $\frac{1}{2}$, $\frac{3}{2}$, ..., $m-\frac{1}{2}$.

Various particular cases are discussed. The method is illustrated by an example.

(T. V. Hanumantha Rao)

2/259

LORD, F. M. (Educational Testing Service, Princeton, N.J.) **2.1** (2.8)

An approach to mental test theory—*In English*

Psychometrika (1959) **24**, 283-302 (37 references, 2 figures)

Five different true-score models are presented. In each an error of measurement is defined as a difference between true-score and observed score, and has an expected value of zero in all circumstances. The models presented then take further assumptions or definitions to show which additional inferences about true-scores can be made. Models discussed are:

the matched-forms,
the rationally equivalent forms,
the item sampling,
the Gaussian errors of measurement,
the binomial error.

The author discusses the various models and the relationships existing between models. He further points out that all five of the true-score models yield good results when applied appropriately. The need is suggested for empirical studies comparing the results obtained under different models to compare their various advantages and disadvantages.

(R. E. Stoltz)

2/260

McFADDEN, J. A. (Purdue University, Indiana) **2.3** (2.7)

Two expansions for the quadrivariate normal integral—*In English*

Biometrika (1960) **47,** 325-333 (24 references)

If X_1, X_2, X_3 and X_4 are normally distributed correlated variables with zero means, the author is concerned to find P_4, the probability that X_1, X_2, X_3 and X_4 are simultaneously positive, in two special cases. For this purpose the generalised tetrachoric series for P_2 [see, for example, Kendall, M. G., *J. R. Statist. Soc.* (1945) **108**, 93-141 and Kendall, M. G. & Stuart, *Advanced Theory of Statistics* (1958), London: Griffin] are not well suited for computation. The two series expansions for P_4 given in this paper do not suffer this disadvantage.

The first case treated is the stationary Markoff case; that is where the inverse of the correlation matrix has zero elements except on the main diagonal and immediately adjacent to it. A series expansion is given which, when the correlation coefficients are positive, gives a good approximation to P_4.

The second case is when $\rho_{13} = \rho_{14} + \rho_{24} = 0$. A series expansion is obtained which gives reasonable results provided none of the correlations is large. The connection between the two series is discussed, and it is shown that the solutions are connected by a simple relationship.

(Florence N. David)

2/261

RAJA RAO, B. (University of Poona) **2.0** (-.-)

Properties of the invariant I_m (m-odd) for distributions admitting sufficient statistics—*In English*

Sankhyā (1959) **21,** 355-362 (6 references)

If f and f' are the density functions for two alternative probability distributions of a random variable X then the expression

$$I_m = \int |f'^{1/m} - f^{1/m}|^m \, dx$$

has been defined by Jeffreys [*Proc. Roy. Soc.* A (1946) **186**, 453-461 and *Theory of Probability* (1948) Oxford: Clarendon Press] as a measure of the distance between the two probability distributions. In the case where f and f' are densities for two members of a Koopman family of distributions, Huzurbazar [*Biometrika* (1955) **42**, 533-537] has given the exact form of I_m for even values of m as an explicit function of the parameters.

The author shows that for odd values of m, I_m can be expressed as

$$\int_{-\infty}^{\infty} (f'^{1/m} - f^{1/m})^m dx - 2 \int_{\{x : f' \leqq f\}} (f'^{1/m} - f^{1/m})^m dx.$$

If f and f' are densities for two members of a Koopman family of distributions, the first integral in the above expression is resolved as an explicit function of the parameters.

The author has also shown this to be the case with the second integral for the following families:

(i) Normal distributions with the parameters varying separately and simultaneously,
(ii) Gamma distributions with the parameters varying simultaneously,
(iii) Poisson distributions, and
(iv) Binomial distributions.

It is noted that the differential forms of I_1 for cases (iii) and (iv) do not lead to the *a priori* probability distributions of the relevant parameters used by Jeffreys in connection with problems of inference regarding these parameters. The author has shown however, that the mth root of the differential forms of I_m for case (i) with the parameters varying separately, do lead to the *a priori* probability distributions of Jeffreys for these parameters.

(P. K. Bhattacharya)

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RUBEN, H. (Columbia University, New York) **2.3** (1.1)

Probability content of regions under spherical normal distributions. I.—*In English*

Ann. Math. Statist. (1960) **31**, 598-618 (40 references)

This is the first of a series of papers attempting to lay the groundwork for a relatively well-rounded theory of the spherical normal distribution. Many distributional problems in the theory of mathematical statistics may be regarded as particular instances of the general problem of determining the probability content of geometrically well-defined regions in Euclidean N-space when the underlying distribution is centered spherical normal and has unit variance in any direction. Specifically it is required that for a definite region R

$$\Pr(R) = (2\pi)^{-\frac{1}{2}N} \int_{x \in R} e^{-\frac{1}{2}x'x} dx$$

in which $x' = (x_1, \ldots, x_N)$, though integrals of this form are rarely capable of being expressed in closed form using well-known functions. It is hoped by the author that this paper will provide a unifying thread and thereby help stimulate further research.

The opening discussion reviews the methods of expressing the spherical normal distribution. In the remainder of the paper the author reviews and discusses a number of important distributional problems which are formally reducible to integral form. The geometric forms which are considered are:

(i) Probability content of a half space,
(ii) Probability contents of centrally and non-centrally located hyperspheres,
(iii) Probability content of a symmetrically and asymmetrically located hyperspherical cone,
(iv) Probability content of a region bounded by a variety of revolution of dimensionality $N-1$ and species p,
(v) Probability contents of symmetrically and asymmetically located hyperspherical cylinders,
(vi) Probability content of a centrally situated ellipsoid,
(vii) Probability content of a regular simplex.

The "method of sections," which will be used frequently to deal with integrals in the sequel paper, is introduced. This consists of dividing up the definite region R by means of a series of parallel and adjoining

continued

RUBEN, H. (Columbia University, New York) **2.3** (1.1)

Probability content of regions under spherical normal distributions. I.—*In English* *continued*

Ann. Math. Statist. (1960) **31**, 598-618 (40 references)

$(N-1)$-flats and of the exploitation of the following fundamental property of the spherical normal distribution of dimensionality N: The conditional probability distribution in any linear subspace of dimensionality $N-k$ $(k = 1, 2, \ldots, N-1)$ is itself spherical normal with dimensionality $N-k$ and with variance in any direction equal to the variance of the original N-dimensional distribution. It follows that the probability content of the infinitesimal region intercepted by R between two parallel infinitesimally spaced flats requires integration over one dimension of a spherical normal distribution which has been integrated over $N-1$ dimensions. If, in addition, the section of each cutting flat is a region of the same geometrical type as R (for example, R an ellipsoid and the section an ellipsoid), with certain geometrical relationships being retained by the $N-1$ dimensional figure, then the probability content integral becomes an integral recurrence relationship.

(D. A. Roellke)

ROGOZIN, B. A. **2.1** (1.5)

A remark on the paper " A moment inequality with an application to the central limit theorem " by C. G. Esseen—*In Russian*

Teor. Veroyat. Primen (1960) **5**, 125-128 (4 references)

It is proved that

$$\lim_{n\to\infty} \inf_{\substack{|a|<\infty \\ 0<\sigma<\infty}} \sup_{x} \sqrt{n} \left| F_n(x) - \mathrm{F}\left(\frac{x-a}{\sigma}\right) \right| < \frac{\rho_3}{\sqrt{2\pi}},$$

where $\mathrm{F}(x)$ is normal distribution function and $F_n(x)$ is a distribution function of a normalised sum of independent identically distributed random variables, $\rho_3 = \dfrac{M\,|\,\xi_i - M\xi_i\,|^3}{\sigma^3}$. The constant $1/(2\pi)^{\frac{1}{2}}$ cannot be improved.

(B. A. Rogozin)

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SENTIS, P. **2.0** (-.-)

Frequency distributions and some of their applications—*In French*

Publ. Inst. Statist. Paris (1958) **7**, 17-91, 93-160 (7 references)

The theory of random variables is well known if one possesses a denumerable infinity of random independent variables. This paper tries to provide some approach to the general case.

In the first chapter the author supposes that all the variables which have the same fixed value can be expressed as certain functions of a set of variates for whom one can fix arbitrary values. The second chapter poses a situation in which the values of the variates are known and lend themselves to a statistical study.

The third and fourth parts are interesting from the point of view of the calculation of the properties when one is able to define the σ functions, additive and π-multiplications. It also generalises the concept of the series and of infinite products in the aggregate of indices which one is unable to count.

In this present work the author introduces a new concept of the multiplication of a variate by means of a number, this allows the definition of the integral function of independent random values, and then the derivation of a Weiner-Lévy process. The generalised infinite products are known to be similar to the exponential of an integral. That which is defined here enables a direct study of the known products of the matrices, of a nucleus integral for example.

It is clear from parts five and six of this paper that the preceeding theories have numerous applications in the field of the calculation of probabilities and the statistic fits the Kolmogoroff equation tolerably well. The author illustrates his paper by practical problems which occur in certain paper factories.

(Mlle. Gervaise)

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TEICHER, H. (Purdue University, Indiana) **2.8** (1.5)

On the mixture of distributions—*In English*

Ann. Math. Statist. (1960) **31**, 55-73 (20 references)

Let $\mathscr{F}=\{F\}$ be a family of distribution functions and let $\{\mu\}$ be a class of measures defined on a Borel field of subsets of $\mathscr{F}$ such that $\mu(\mathscr{F})=1$ for every μ of $\{\mu\}$. The distribution function

$$H=H(x)=\int_{\mathscr{F}} F(x)d\mu(F)$$

is called the μ-mixture of $\mathscr{F}$. If μ varies over the class $\{\mu\}$ then one obtains the class $\{H\}$ of mixtures H. In this paper the author considers families $\mathscr{F}=\{F(x; \alpha_1, \ldots, \alpha_m)\}$ which depend on a finite number of parameters. He treats

(i) Convergence questions which occur when either of the sequences F_n or μ_k tend (in the sense of weak convergence) to a limit.

(ii) Mixture of additively closed families.

Mixtures of additively closed families are shown to be closed under convolutions. Conditions which govern normality are given for the μ-mixtures of normal distributions. It is also shown that the mixture of symmetric stable distributions with fixed exponent $\alpha(0<\alpha\leqq 2)$ cannot be a symmetric stable distribution with exponent α. Similarly, the mixture of normal distributions with identical means cannot be normal. The author presents a number of complex theorems.

(E. Lukács)

ZELEN, M. & SEVERO, N. C. (Nat. Bur. Standards, Washington, D.C.) **2.3** (11.1)

Graphs for bivariate normal probabilities—*In English*

Ann. Math. Statist. (1960) **31**, 619-624 (5 references, 3 figures)

In presenting this paper of three charts which will enable one to compute easily the bivariate normal integral to a maximum error of 1/100, the author states that this work was motivated by the recent work by Owen & Wisen on this subject and also the fact that there has been much activity recently dealing with the tabulation of the bivariate normal probability integral: see Owen & Wiesen " A method of computing bivariate normal probabilities " [*Bell System Tech. J.* (1959) **38**, 1-20] and the National Bureau of Standards, " Tables of the bivariate normal distribution functions and related functions " [*N.B.S., Applied Maths. Series* No. 50 (1959) U.S. Government Printing Office].

The authors first discuss notation and useful formulas relating to the bivariate normal integral: it is then shown (following the work of Owen) that this integral may be expressed as a sum of the functions of the two parameters only. This two-parameter function is then plotted by exhibiting its level curves in steps of 0·01.

The three charts do not cover the complete range of parameter values, but simple transformations are available for extending the given range.

The application of the charts, which includes visual interpretation, is illustrated by three examples including one which involves a non-rectangular region.

(D. H. Shaffer)

HARTER, H. L. (Aero. Res. Labs., Dayton, Ohio) **3.8** (11.1)

Expected values of normal order statistics—*In English*

ARL Tech. Rep. 60-292 (1960), Wright-Patterson Air Force Base. xvi+32 pp. (18 references, 3 tables)

A brief history is given of the development of the theory of order statistics since the work of Karl Pearson [*Biometrika* (1902) **1**, 390-399], and of past efforts to tabulate their expected values for samples from a normal population, beginning with a table of the mean range of samples of size $n = 2(1)1000$, published by Tippett [*Biometrika* (1925) **17**, 364-387]. A fuller account is given of the method of computation of a new five-decimal place table of the expected values of all order statistics for samples of size n from a normal population. The expected value of the kth largest observation in a sample of size n from a standard normal population (mean zero and variance one) is given by the equation

$$\mathscr{E}(x_{k\,|\,n}) = \frac{n!}{(n-k)!(k-1)!} \int_{-\infty}^{\infty} x[\tfrac{1}{2}-\Phi(x)]^{k-1}[\tfrac{1}{2}+\Phi(x)]^{n-k}\phi(x)dx,$$

where $\phi(x) = (2\pi)^{-\frac{1}{2}}e^{-x^2/2}$ and $\Phi(x) = \int_0^x \phi(x)dx$.

The expected value of the kth smallest observation is given by the same expression preceded by a minus sign; so that for a given value of n it is necessary to compute the expected values only for $k = 1(1)[n/2]$. This was done by numerical integration on the Univac Scientific (ERA 1103A) computer, for $n = 2(1)100$ and for values of n none of whose prime factors exceeds seven, up through $n = 400$. The resulting table, rounded to five decimal places, and accurate to within a unit in the last place, is included in the report.

A discussion is given of a proposal by Blom [*Statistical Estimates and Transformed Beta-Variables* (1958), New York: Wiley] for approximating the ith normal order statistic (ith smallest normal deviate for a sample of size n) by means of the relation

$$\mathscr{E}(x_i) = \Phi^{-1}\left(\frac{i-\alpha}{n-2\alpha+1}\right),$$

where $\Phi(x) = \int_{-\infty}^{x} \phi(x)dx$, with $\phi(x) = \{1/(2\pi)^{\frac{1}{2}}\}\{1/e^{\frac{1}{2}x^2}\}$.

Blom tabulated the value of α required to yield the correct value of $\mathscr{E}(x_i)$ for $i = 1(1)[n/2]$ when $n = 2(2)10(5)20$, and suggested the use of $\alpha = 3/8$ as a compromise value. Table II of the present report extends

continued

HARTER, H. L. (Aero. Res. Labs., Dayton, Ohio) **3.8** (11.1)

Expected values of normal order statistics—*In English*

continued

ARL Tech. Rep. 60-292 (1960), Wright-Patterson Air Force Base. xvi+32 pp. (18 references, 3 tables)

these results to include $i = 1(1)25(5)100(10)[n/2]$ for $n = 25$, 50, 100, 200 and 400. When n is of this magnitude, the value 3/8 for α is too small. Since α is quite sensitive to changes in n, no single value of α is satisfactory for all values of n; it is possible, however, to do a fairly good job of estimating $\mathscr{E}(x_i)$ by choosing one or two compromise values of α for each n. Such values are given in Table III for $n = 2(2)10(5)25$, 50, 100, 200 and 400, along with regression equations to be used for intermediate values of n, and a discussion of the errors involved in these approximations.

A discussion is given of actual and potential uses of the tables, foremost of which is transformation to standard normal scores of ranked data or of other data not conforming to the assumptions underlying the analysis of variance.

H. L. Harter

HARTER, H. L. & CLEMM, D. S. (Wright Air Development Center, Dayton, Ohio) **3.8** (11.1)

The probability integrals of the range and of the Studentised range-probability integral, percentage points, and moments of the range—*In English*

WADC Technical Report 58-484, I (1959) Aero. Res. Labs., Wright-Patterson Air Force Base. xii+139 pp. (19 references, 3 tables)

A description is given of the computation and use of tables of the probability integral of the range, percentage points of the range, and moments of the range for samples from a normal distribution, all most extensive and more accurate than previously published tables. The introductory material includes a brief summary of the development of the theory and the computation of tables of the range as far back as the work of Tippett [*Biometrika* (1925), **17**, 364-387], also an account of applications dating back to the demonstration by E. S. Pearson [*British Standards Institution* No. 600, (1935)] that the range is useful in statistical quality control. It is stated that the principal reason for computing the new table of the probability integral of the range was the desire to use it in computing more accurate tables of the probability integral and of the percentage points of the studentised range (see Volume II of this report; abstracted in this journal No. **2**/273, **3.8**). The method of computation of the tables on the Univac Scientific (ERA 1103 and 1103A) computer is discussed, along with the accuracy of the results. The probability integral and the moments of the range were obtained by numerical integration, employing the seven-point Lagrangian integration formula, and using double-precision arithmetic in the case of the moments. The percentage points of the range were obtained by inverse interpolation in the table of the probability integral. In the section on the use of the tables, an example is given of the use of percentage points of the range in tests of hypotheses concerning the standard deviation of a normal population. The table of moments of the range, like that of the probability integral, is useful primarily in computing other tables.

The following tables are included: (1) an eight-decimal-place table of the probability integral of the (standardised) range, $W = w/\sigma$, at intervals of 0·01 for samples of size $n = 2(1)20(2)40(10)100$; (2) a six-decimal-place table of percentage points of the range for the same values of n and cumulative probability $P = 0{\cdot}0001$, 0·0005, 0·001, 0·005, 0·01, 0·025, 0·05,

continued

HARTER, H. L. & CLEMM, D. S. (Wright Air Development Center, Dayton, Ohio) **3.8** (11.1)

The probability integrals of the range and of the Studentised range-probability integral, percentage points, and moments of the range—*In English*

WADC Technical Report 58-484, I (1959) Aero. Res. Labs., Wright-Patterson Air Force Base. xii+139 pp. (19 references, 3 tables)

continued

0·1(0·1)0·9, 0·95, 0·975, 0·99, 0·995, 0·999, 0·9995 and 0·9999; and (3) a table of moments of the range [mean to 10 decimal places (11 significant figures), variance to 10 decimal places (10 significant figures), skewness to 8 decimal places (8 significant figures) and elongation (sometimes inappropriately called kurtosis) to 7 decimal places (8 significant figures)] for samples of size $n = 2(1)100$.

(H. L. Harter)

HARTER, H. L., CLEMM, D. S. & GUTHRIE, E. H. (Wright Air Development Center, Dayton, Ohio) **3.8** (11.1)

The probability integrals of the range and of the Studentised range—probability integral and percentage points of the Studentised range; critical values for Duncan's new multiple range test—*In English*

WADC Technical Report 58-484, II (1959) Aero. Res. Labs., Wright-Patterson Air Force Base. xxi+293 pp. (15 references, 3 tables)

A description is given of the computation and use of tables of the probability integral of the studentised range, percentage points of the studentised range, and critical values for Duncan's new multiple range test, all based upon samples from a normal distribution, and all more extensive and more accurate than previously published tables. The introductory material includes a brief summary of the development of the theory and the computation of tables of the studentised range from the time (1932) that "Student" (W. S. Gosset) first proposed the studentisation of the range. It is stated that the primary reason for computing the new table of the probability integral of the studentised range was the desire to use it in computing more extensive and more accurate tables of percentage points. More accurate tables were needed especially in the case of the special percentage points, for protection levels based upon degrees-of-freedom, which are the critical values for Duncan's new multiple-range test, for which previously published tables contain sizeable inaccuracies. The method of computation of the tables on the Univac Scientific (ERA 1103 and 1103A) computer is discussed, along with the accuracy of the results. The probability integral of the studentised range was obtained by numerical integration, employing the seven-point Lagrangian integration formula and using a new 8-place table of the probability integral of the range [see Volume I of this report: abstracted in this journal No. **2**/271, **3.8**]. The regular percentage points and the special percentage points which are the critical values for Duncan's test were obtained by inverse interpolation in the table of the probability integral of the studentised range. An example illustrates the use of the tables in performing two multiple comparisons tests on observed

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continued

HARTER, H. L., CLEMM, D. S. & GUTHRIE, E. H. (Wright Air Development Center, Dayton, Ohio) **3.8** (11.1) *continued*

The probability integrals of the range and of the Studentised range—probability integral and percentage points of the Studentised range; critical values for Duncan's new multiple range test—*In English*

WADC Technical Report 58-484, II (1959) Aero. Res. Labs., Wright-Patterson Air Force Base. xxi+293 pp. (15 references, 3 tables)

data, the regular percentage points being used for the Newman-Keuls test and the special percentage points for Duncan's test.

The following tables are included:

(i) a table (to six decimal places or six significant figures, whichever is less accurate) of the probability integral of the studentised range, $Q = w/s$, with varying intervals (small enough to make the table interpolable) for samples of size $n = 2(1)20(2)40(10)100$, with degrees of freedom $\nu = 1(1)20$, 24, 30, 40, 60, and 120 for the independent estimate s of the population standard deviation;

(ii) a table (to four decimal places or four significant figures, whichever is less accurate) of the percentage points of the studentised range corresponding to cumulative probabilities $P = 0{\cdot}001$, 0·005, 0·01, 0·025, 0·05, 0·1(0·1)0·9, 0·95, 0·975, 0·99, 0·995, and 0·999 for the same values of n and ν, also for $\nu = \infty$; and

(iii) a table (to four decimal places or four significant figures, whichever is less accurate) of the critical values for Duncan's new multiple range test at significance levels $\alpha = 0{\cdot}1$, 0·05, 0·01, 0·05, and 0·001, corresponding respectively to protection levels $P = (0{\cdot}9)^{n-1}, (0{\cdot}95)^{n-1}, (0{\cdot}99)^{n-1}, (0{\cdot}995)^{n-1}$, and $(0{\cdot}999)^{n-1}$, for the same values of n and ν (including $\nu = \infty$).

(H. L. Harter)

BARTON, D. E., DAVID, Florence N. & O'NEILL, Anne F. (University Coll., London and Wheaton Coll., Norton, U.S.A.) **3.2** (2.7)

Some properties of the distribution of the logarithm of non-central F—*In English*

Biometrika (1960) **47**, 417-424 (11 references, 4 tables)

Fisher & Cornish [*Rev. Int. Statist. Inst.* (1937) **5**, 307-321] showed that the distribution of the logarithm of central F can be approximated to be means of an Edgeworth expansion.

The authors consider the logarithm of the non-central F, where non-central F is defined as the ratio of χ'^2/f_1 and χ^2/f_2 where χ'^2 is a non-central χ^2 with f_1 degrees of freedom and parameter λ and χ^2 is an independent χ^2 with f_2 degrees of freedom. This logarithm they denote by z^* and, following the procedure of Fisher & Cornish, they discuss expansions based on the assumption that an Edgeworth expansion is adequate. The moments of z^* are given in terms of the polygamma functions plus a simple series expansion which is quickly convergent.

Two series expansions are found to be of value. First, for λ small, the distribution of z^* is expanded about the appropriate point of the central z. Secondly, the Edgeworth expansion for z^* is given. Since as many moments of z^* can be obtained as desired, the possibility of fitting a Johnson S_u curve is also considered.

Tables are given whereby the moments of z^* can easily be evaluated. Comparisons are made between the several approximations suggested: [see also abstracts No. **2**/278, **3.2** and No. **2**/283, **3.2** in this present journal].

(Florence N. David)

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KHATRI, C. G. (University of Baroda) **3.9** (0.6)

On conditions for the forms of the type XAX' to be distributed independently or to obey Wishart distribution—*In English*

Bull. Calcutta Statist. Ass. (1959) **8**, 162-168 (4 references)

Let X be a $p \times n$ random matrix formed by independently distributed column vectors. Let the jth column vector have the multivariate normal distribution with mean vector μ_j and dispersion matrix Σ. What are the conditions that a $n \times n$ symmetric matrix A should satisfy in order that XAX' should have the noncentral Wishart distribution? The author proves that if A is a matrix whose rank is greater than p, then a necessary and sufficient condition is that A should be idempotent.

A second problem which the author considers is the following: what are the conditions which the two $n \times n$ symmetric matrices A and B should satisfy in order that the forms XAX' and XBX' should be distributed independently of each other? The author shows that a necessary and sufficient condition for this is that $AB = 0$. From this the following corollaries are deduced:

(i) Let A be an $n \times n$ matrix and L an $n \times t$ matrix. A necessary and sufficient condition that XAX' and XL are independently distributed is that $AL = 0$;

(ii) Let A_i ($i = 1, 2, ..., m$) be m, $n \times n$ symmetric matrices. A necessary and sufficient condition that the forms XA_iX' ($i = 1, 2, ..., m$) be independently distributed is that $A_iA_j = 0$ ($i \neq j$).

Using the results mentioned above the author also obtains theorems similar to Cochran's theorem on the distribution of quadratic forms.

Finally the following results concerning the distribution of $X'AX$ (as distinct from XAX') are stated without proof.

(i) Let A be a $p \times p$ matrix of rank greater than n. A necessary and sufficient condition for $X'AX$ to be Wishart is that $A\Sigma A = A$ that is, $A\Sigma$ should be idempotent.

(ii) A necessary and sufficient condition for $X'AX$ and $X'BX$ to be independently distributed is that $A\Sigma B = 0$.

(S. John)

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LAHA, R. G. & LUKACS, E. (Catholic University of America, Washington, D.C.) **3.4** (0.6)
On certain functions of normal variates which are uncorrelated of high order—*In English*
Biometrika (1960) **47**, 175-176 (5 references)

Necessary and sufficient conditions are given such that an inhomogeneous quadratic form in n independent random normal variables shall be independent of a linear form in these same variables. Thus if $Q = \boldsymbol{x}\boldsymbol{A}\boldsymbol{x}' + \boldsymbol{b}\boldsymbol{x}'$ and $L = \boldsymbol{c}\boldsymbol{x}'$ are the quadratic and linear forms respectively where $\boldsymbol{x}$ is a row vector of n independent normal variables and $\boldsymbol{A}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are matrices of constant coefficients, then Q and L are independent if and only if $\boldsymbol{c}\boldsymbol{A} = 0$ and $\boldsymbol{c}\boldsymbol{b}' = 0$. If $\mathscr{E}(Q^iL^j) = \mathscr{E}(Q^i)\mathscr{E}(L^j)$, $(i = 1, \ldots, r;\ j = 1, \ldots, s)$ Q and L are said to be uncorrelated to order (r, s). It is shown that if Q and L are uncorrelated to order (2,2) then $\boldsymbol{c}\boldsymbol{A} = \boldsymbol{c}\boldsymbol{b}' = 0$. If Q is homogeneous quadratic, then the conditions for independence derived by Laha & Lukács are weaker than those conditions which may be derived from a result by Kawada [*Ann. Math. Statist.* (1950) **21**, 614-615] which require that Q and L shall be uncorrelated to order (2,4).

(J. G. Saw)

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PEARSON, E. S. (University College, London) **3.2** (–.–)
Editorial Note on non-central F approximations—*In English*
Biometrika (1960) **47**, 430-431 (1 reference, 1 table)

The author, the Editor of *Biometrika*, compares and contrasts the approximations to the probability integral of non-central F suggested by Severo & Zelen [*Biometrika* (1960) **47**, 411-416: abstracted in this present journal No. **2**/283, **3.2**] with those put forward by Barton, David, F. N. & O'Neill [*Biometrika* (1960) **47**, 417-429: abstracted in this present journal No. **2**/275, **3.2**].

He shows that, over a wide range of values of f_1, f_2 and λ, there is little to choose between the two sets of approximations. Further, it appears that the results obtained by using a Johnson S_u curve [*Biometrika* (1949) **36**, 149-176] are on the whole better than either set.

(Florence N. David)

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PRANAB, K. S. (Calcutta University) **3.8** (1.5)

On the moments of sample quantiles—*In English*

Bull. Calcutta Statist. Ass. (1959) **9**, 1-19 (3 references)

This paper deals with the existence of moments of the ordered statistics and the convergence of the moments of the sample quantiles as the sample size tends to infinity.

The rth ordered statistic (the rth smallest observation) in a sample of size n is denoted by $x_{(r)}$. The pth sample quantile is defined to be the ordered statistic $x_{([np]+1)}$ and the pth quantile of the population is denoted by ξ_p. Throughout this paper it is assumed that the population distribution function $F(x)$ admits of a continuous density function $f(x)$. The main results of this paper are the following:

Theorem 1. If the population admits of the δth moment with $\delta>0$ then the kth moment of the rth ordered statistic exists whenever r satisfies the inequality $r_0 \leqq r \leqq n-r_0-1$ where $\delta r = k$. A consequence of Theorem 1 is that the kth moment of a sample quantile will exist in sufficiently large samples.

It is known that if $f(x)$ is non-zero at ξ_p, then the sample pth quantile has an asymptotic normal distribution. Concerning the convergence of moments the author states the following theorem:

Theorem 2. Let

(i) the population admit of a δth moment for some $\delta>0$,

(ii) $f(x)$ be non-zero at ξ_p and

(iii) $f(x)$ have a bounded derivative in a neighbourhood of ξ_p:

then the moments of the standardised sample pth quantile converge to those of the normal distributions. Theorem 2 is a generalisation of a result due to Chu & Hotelling [*Ann. Math. Statist.* (1955) **26**, 593-606].

In the concluding section the author indicates how, in particular situations, the conditions can be relaxed and states, without proving, the following corollary:

Corollary 1. For any distribution with a continuous density function and finite δth moment with $\delta>0$, the central moments of the pth sample quantile converge to zero provided the population pth quantile is uniquely defined.

(J. Sethuraman)

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RIDER, P. R. (Wright-Patterson Air Force Base, Dayton, Ohio) **3.6** (3.8)

Variance of the median of small samples from several special populations—*In English*

J. Amer. Statist. Ass. (1960) **55**, 148-150 (4 references, 7 tables)

The distribution of the median of a sample of size n from a population having the density function $f(x)$ is asymptotically normal with mean m and variance given by the reciprocal of $4n[f(m)]^2$, where m is the population median. This asymptotic variance is compared with the exact variance of the median for small samples from several special populations.

The populations considered were exponential, normal, rectangular, cosine, parabolic and inverted parabolic. Samples of size $n = 1, 3, 5, 7$ were taken for comparison. For each population and for each sample size a table of exact variance, asymptotic variance and relative error is given.

It is found that for each population the error decreases monotonically as the sample size increases. The relative error decreases among the six populations as the standardised fourth moment increases, so that the asymptotic variance is most accurate for the exponential population, followed in order by normal, cosine, parabolic, rectangular and inverted parabolic. Finally, it is found that the asymptotic variance exceeds the exact variance except for the exponential population.

(S. Krane)

ROY, A. R. & MOHANTY, S. G. (Indian Council of Agric. Res., New Delhi) **3.1** (5.6)

Distribution of χ^2-analogue for normal population with class intervals defined in terms of the sample median—*In English*

J. Indian Soc. Agric. Statist. (1958) **10**, 90-98 (4 references)

The authors consider independent observations $x_1, x_2, \ldots, x_n$ from $N(\theta, 1)$ and the goodness-of-fit chi-square statistics

$$R_n = \sum_1^3 (O_i - E_i)/E_i$$

where, 0_1, 0_2 and 0_3 are the frequencies of observations in intervals $(-\infty, \tilde{x}+a]$, $(\tilde{x}+a, \tilde{x}+b)$, $[\tilde{x}+b, \infty)$; $\tilde{x}$ is the sample median, a and b are constants $(a<b)$, $E_i = n\pi_i$ $(i = 1, 2, 3)$; and

$$\pi_1 = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{a} \exp(-\tfrac{1}{2}t^2)dt,$$

$$\pi_2 = (2\pi)^{-\frac{1}{2}} \int_{a}^{b} \exp(-\tfrac{1}{2}t^2)dt,$$

$$\pi_3 = 1/\sqrt{2\pi} \int_{b}^{\infty} \exp(-\tfrac{1}{2}t^2)dt.$$

It is seen that E_1, E_2 and E_3 are expected frequencies in the non-stochastic intervals $(-\infty, \theta+a]$, $(\theta+a, \theta+b)$, $[\theta+b, \infty)$. The authors show that in the limit, as $n\to\infty$, R_n is distributed in the same way as $\lambda_1 y_1^2 + \lambda_2 y_2^2$ where y_1 and y_2 are independent standard normal variables and λ_1, λ_2 are constants depending on a, b but not on θ. For each a, b $(a<b)$, $\lambda_1<1$ and $\lambda_2\geqq 1$, the equality sign holding only when $a+b = 1$.

The proof runs as follows: consider the intervals $(-\infty, \theta+a]$, $(\theta+a, \theta+b)$, $[\theta+b, \infty)$ and define $b_i(x)$ $(i = 1, 2, 3)$ as the function which takes the value one if x is in the ith interval and otherwise the value zero. It is demonstrated that for each $i = 1, 2, 3$,

$$(1/\sqrt{n})(O_i - E_i) = \sqrt{n}[U_i + c_i(\tilde{x}-\theta)] + o_p(1)$$

where $U_i = (1/n) \sum_{j=1}^{n} [b_i(x_j) - \pi_i]$

$$c_1 = (2\pi)^{-\frac{1}{2}} e^{-a^2/2}, \quad c_2 = (2\pi)^{-\frac{1}{2}}[e^{-b^2/2} - e^{-a^2/2}],$$

$$c_3 = -(2\pi)^{-\frac{1}{2}} e^{-b^2/2}.$$

This makes use of the limiting multivariate normal distribution of the median and U-statistics obtained by Sukhatme [*J. R. Statist. Soc.* B (1957) **19**, 144-148], to establish a similar property of $(O_i - E_i)$ $(i = 1, 2, 3)$. The main bulk of the algebra is in the computation of the asymptotic moment-matrix of the variables $E_i^{\frac{1}{2}}(O_i - E_i)$ and in deducing the properties of its two non-zero latent roots λ_1 and λ_2.

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(S. K. Mitra)

SETHURAMAN, J. & SUKHATME, B. V. (Indian Statist. Inst., Calcutta and Indian Coun. Agric. Res., New Delhi) **3.8** (3.9)

Joint asymptotic distribution of U-statistics and order statistics—*In English*

Sankhyā (1959) **21**, 289-298 (3 references)

It is shown under some mild restrictions that the joint distribution of a U-statistic, (see Hoeffding [*Ann. Math. Statist.* (1948) **19**, 293-325]), and the a_nth order statistic, computed from n independent observations on a random variable X, both suitably standardised, tends to

(i) the bivariate normal distribution, if $a_n/n\to p$, $0<p<1$;

(ii) the joint distribution of two independent variables one of which is gamma and the other normal, if $a_n\to K$ or $n-a_n\to K$ where K is a constant not depending on n;

(iii) the joint distribution of two independent normal variables, if $a_n\to\infty$ such that $a_n/n\to 0$ or 1.

The proof requires the following assumptions:

(a) The U-statistic is generated from a bounded symmetric kernel.

(b) The random variable X has a continuous distribution function $F(x)$, and in case (i) it is further assumed that

(c) F has a density which is continuous at the pth quartile.

An earlier paper by one of the authors, Sukhatme [*J. R. Statist. Soc.* B (1957) **19**, 144-148], proves case (i) for the special value of $p = 1/2$. The proof in the present paper proceeds on similar lines and consists mainly in evaluating the limiting expression for the characteristic function of the two standardised statistics.

It is stated that in proving case (i) the assumption (a) could be somewhat relaxed and replaced by the following weaker assumptions.

(a1) The U-statistic is generated from a symmetric kernel which is bounded on every bounded interval of its arguments.

(a2) The kernel considered as a random variable with its arguments replaced by independent observations on X, has a finite third moment.

The results (i)-(iii) are generalised several U-statistics and several order statistics: analogous results are true for the generalised U-statistics of Lehmann [*Ann. Math. Statist.* (1951) **22**, 165-179] computed from independent observations on several random variables.

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(S. K. Mitra)

SEVERO, N. C. & ZELEN, M. (University of Buffalo and National Bureau of Standards) **3.2** (3.3)

Normal approximation to the chi-squared and non-central F probability functions—*In English*

Biometrika (1960) **47**, 411-416 (7 references, 5 tables)

The authors derive an approximation to the probability integral of the χ^2 distribution based on the Wilson-Hilferty cube root transformation. Tables are given of the various factors which intervene in this approximation.

The non-central F may be defined as $(\chi'^2/f_1)/(\chi_2^2/f_2)$ where $\chi_1'^2$ is a non-central χ^2 with f_1 degrees of freedom and parameter λ, and χ_2^2 is an independent central χ^2 with f_2 degrees of freedom. Abdel-Aty [*Biometrika* (1954) **41**, 538-542] suggested that $(\chi_2'^2/f_1+\lambda)^{\frac{1}{3}} = Y_1'$ might be approximately normal and this is the result which the authors use.

Writing Y_2^3 for the denominator of the non-central F and c for the cube root of $(f_1+\lambda)/f_2$, the authors remark, following the procedure suggested by David & Johnson [*Biometrika* (1951) **38**, 43-57], that a probability statement about non-central F can be reduced to finding the probability that $L = cY_1' - (F')^{\frac{1}{3}}Y_2$ is less than or equal to zero. L is approximately normal with mean (μ_2) and variance (σ_L) deducable from Abdel-Aty's work and $\Pr\{L \leqq O\} \fallingdotseq \Pr(-\mu_2/\sigma_L)$. Consequently estimation of the non-central F distribution can be thrown back to tables of the normal integral.

Some comparison is given with the exact values of the probability integral calculated by Tang [*Stat. Res. Mem.* (1938) **2**, 126-149] and with the approximation put forward by Patnaik [*Biometrika* (1949) **36**, 202-232]. It is clear that the approximation suggested by the authors is good over a wide range of values of f_1, f_2 and λ.

Further comparison are given by E. S. Pearson [*Biometrika* (1960) **47**, 430-431: abstracted in this present journal No. **2**/278, **3.2**]; see also Barton, David F. N. & O'Neill [*Biometrika* (1960) **47**, 417-429: abstracted in this present journal No. **2**/275, **3.2**].

(Florence N. David)

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SREENIVASA IYENGAR, N. (Indian Statist. Inst. Calcutta) **3.7** (4.3)

On the standard error of the Lorentz concentration—*In English*

Sankhyā (1960) **22**, 371-378 (2 references)

In this paper the standard error of the Lorentz concentration ratio has been estimated on the assumption of the lognormality of the distribution of incomes. Exact and large sample tests are then suggested for testing the equality of several concentration ratios.

If $\log x$ has a normal distribution with mean θ and standard deviation λ, then the concentration ratio L, defined as twice the area between the egalitarian line and the Lorentz curve is given by

$$L = 2\phi(x/\sqrt{2}),\ \phi(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\xi} e^{-\frac{1}{2}t^2}dt.$$

L is independent of the location parameter θ. Substituting $\hat{\lambda}$, the maximum likelihood estimate of λ in the above equation, the maximum likelihood estimate $\hat{L}$ of L is obtained. $\hat{L}$ has an asymptotic normal distribution with mean L and standard error $(2\pi n)^{-\frac{1}{2}}\exp(-\frac{1}{4}\lambda^2)$ where n is the number of observations. Substituting $\hat{\lambda}$ for λ, an estimate of the standard error is obtained. Usually, however, only grouped data are available. The maximum likelihood estimate of λ is then tedious to compute. A simpler method based on the empirical Lorentz curve can be adopted. In the latter case, the estimate of the standard error obtained from the above expression will be an under-estimate. If the maximum likelihood method of estimation for grouped data is used, as is the practice in probit analysis, the above will give a close lower bound to the true standard error.

The test for equality of several concentration ratios based on independent samples reduces to Bartlett's test for equality of variances. In the special case of testing equality of two concentration ratios, the variance ratio can be used. If the sample sizes are large, the difference $\log\lambda_1 - \log\lambda_2$ can be tested by the usual normal approximation.

Finally, an empirical investigation is made to test the equality of two Lorentz ratios obtained from two interpenetrating sub-samples of the National Sample Survey, India.

(R. N. Bhattacharyya)

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TURIN, G. L. (Hughes Research Labs., Culver City, California) **3.4** (2.1)

The characteristic function of Hermitian quadratic forms in complex normal variables—*In English*

Biometrika (1960) **47,** 199-201 (9 references)

Wooding [*Biometrika* (1956) **43,** 212-125] has given an expression for the multivariate distribution of several complex normal variables of a special type often encountered in the analysis of random noise. Using Wooding's results the author finds the characteristic function of the real quadratic form in these variables and thence, by a Fourier transformation on the characteristic function, the density function of the real quadratic forms.

The distribution is said to arise in, for example, the computation of the probability that an ideal binary hypothesis-testing communications receiver will err in its decision concerning which of two signals was transmitted through a channel comprising both a random transmission medium, such as the ionosphere, and additive thermal noise.

(J. G. Saw)

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WALLACE, D. L. (University of Chicago) **3.1** (3.3)

Bounds on normal approximations to Student's and the Chi-square distributions—*In English*

Ann. Math. Statist. (1959) **30,** 1121-30 (6 references, 2 tables)

Formulae closely related to

$$u(t) = [n \log_e (1+t^2/n)]^{\frac{1}{2}}$$
$$w(\chi^2) = [\chi^2 - n - n \log_e (\chi^2/n)]^{\frac{1}{2}}$$

are considered for converting upper tail values of Student's t or chi-square variates with n degrees of freedom to normal deviates. Bounds on the exact normal deviate are constructed that are valid for any fixed n uniformly in the entire upper tail. Among the bounds, the following are the simplest. If $x_n(t)$ is the exact normal deviate corresponding to t, then for any $n \geqq 1$

$$u(t)[1-(0{\cdot}293/n)] \leqq x_n(t) \leqq u(t)$$
$$u(t)-(0{\cdot}368/\sqrt{n}) \leqq x_n(t).$$

If $y_n(\chi^2)$ is the exact normal deviate corresponding to χ^2, then for any $n \geqq 1$, and any $\chi^2 > n$

$$w(\chi^2) \leqq y_n(\chi^2) \leqq w(\chi^2) + (0{\cdot}60/\sqrt{n}).$$

Closely related to the bounds for Student's t is the approximation

$$\left(\frac{8n+1}{8n+3}\right) u(t)$$

to the normal deviate $x_n(t)$. It is claimed to be accurate to within 0·02 for $t^2/n < 5$ and to be superior to common approximations (even to the Paulson approximation). A more complex approximation with wider range of accuracy is given.

For chi-square, the Wilson-Hilferty approximation is much superior to the bounds as approximations except in the extreme tail and the chief value of the approximation is the uniform bound of order $\sqrt{n}$ on the error in the tail which is useful in theoretical work.

(D. L. Wallace)

COHEN, A. C., Jr. (University of Georgia, Athens, U.S.A.) **4.3** (2.5)

Estimating the parameters of a modified Poisson distribution—*In English*

J. Amer. Statist. Ass. (1960) **55**, 139-143 (9 references, 1 table)

The estimation of the parameter of a Poisson distribution is considered for the special case in which there exists a (possibly) non-zero probability that any value of " one " is reported as " zero ". The maximum likelihood estimate of the Poisson parameter is found as the unique positive root of a quadratic equation except for degenerate cases. The probability of misreporting " one " as " zero " is also readily estimable by maximum likelihood. Asymptotic variances and covariance of these estimates are obtained by inversion of the information matrix. As an example, the author alters the well-known Bortkiewicz data on deaths from horse kicks in the Prussian army.

(S. Krane)

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CORNISH, E. A. (Div. of Math. Statist., CSIRO, Adelaide) **4.2** (3.9)

The simultaneous fiducial distribution of the parameters of a normal bivariate distribution with equal variances—*In English*

CSIRO Div. of Math. Statist., Tech. Paper, No. 6 (1960) 1-8 (7 references)

In his *Statistical Methods and Scientific Inference* Fisher [(1959), Edinburgh: Oliver & Boyd] has derived the fiducial distribution of the parameters in a bivariate normal population having means ξ_1, ξ_2, variances σ_1^2, σ_2^2 and correlation coefficient ρ. This paper is concerned with the corresponding fiducial distribution for a bivariate normal population with means ξ_1, ξ_2, equal variances σ^2, and correlation coefficient ρ.

After obtaining the distribution of the sample statistics $\bar{x}_1$, $\bar{x}_2$, s, r, the author derives the simultaneous fiducial distribution of ξ_1, ξ_2, ρ, σ.

(J. Gani)

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CZEN, P. & DZAN, D. I. (Inst. Math., Polish Acad. Sci., Wrocław and Wrocław Univ.) **4.0** (8.7)

On the division of the number of observations by using the ratio-delay method—*In English*

Zast. Mat. (1960) **5**, 179-194

If we estimate the ratio of working-time of workers on the basis of n observations taken at random moments and we use Rubin's estimator $\bar{p} = (m+\sqrt{n}/2)/(n+\sqrt{n})$, where m is the number of positive observations, then the mean-square error of estimation does not depend on the true working-time ratio p and we have for the expectation of the square of the error the relation

$$\mathscr{E}(\bar{p}-p)^2 = 1/4(1+\sqrt{n})^2.$$

Imagine that a factory works in k shifts; that we are interested in estimating working-time ratio for each shift separately; that the loss caused by error of estimating working-time ratio is proportional to the square of the error with coefficient c_i for the ith shift and that we can take n observations in all. If we then take n_i observations in the ith shift, $n_1+n_2+\ldots+n_k = n$, then the expected loss will be

$$\mathscr{E}(S) = \sum_{i=1}^{k} \frac{c_i}{4(1+\sqrt{n_i})^2}.$$

The authors discuss the problem of optimal division of the total number of observations between the k shifts in order to minimise the expected loss $\mathscr{E}(S)$. They find an approximate solution of the form $b_i n+a_i$, where the b's add to unity and are proportional to the square-roots of the corresponding c's. The error of this approximation is estimated for $k = 2$ and $k = 3$. Some examples are calculated and compared with the exact solutions.

(S. Zubrzycki)

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DOROGOVTZEV, A. I. (University of Kiev) **4.1** (4.4)

Confidence intervals in estimation of parameters—*In Ukrainian*

Dopov. Acad. Nauk Ukrain. SSR (1959) 355-358 (2 references)

The author considers the expansion of the confidence interval x of the maximum likelihood estimate $\hat{\theta}$ of the parameter θ by $1/\sqrt{n}$ powers (where n is the number of independent observations of a random variable ξ with a distribution function belonging to the family $F(x, \theta)$; that is to say x is found in the form

$$x = x_0 + \frac{x_1}{\sqrt{n}} + \frac{x_2}{n} + \ldots,$$

where x satisfies the correlation

$$\Pr\left\{\sqrt{n}\left|\frac{\theta-\hat{\theta}}{s}\right| < x\right\} = 1-\alpha$$

$(0<\alpha<1)$, where s^2 denotes the variance of $\hat{\theta}$.

(A. I. Dorogovtzev)

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DUNN, Olive J. (University of California, Los Angeles) **4.2** (4.8)

Confidence intervals for the means of dependent, normally distributed random variables—*In English*

J. Amer. Statist. Ass. (1959) **54**, 613-621 (6 references, 4 tables)

The author considers the problem of constructing k simultaneous confidence intervals for the means of k correlated normal variates, based on n (k-variate) observations. Intervals having bounded confidence are given for arbitrary unknown covariances and for variances known, unknown, and unknown but equal. The mathematical development was given in an earlier paper [*Ann. Math. Statist.* (1958) **29**, 1095-1111: abstracted in this journal No. **1**/45, **4.2**].

In the present paper the application of the results is discussed, some tables are given to facilitate calculations, and the methods are illustrated by worked examples. The most generally satisfactory solution from the standpoint of weakness of assumptions and shortness of intervals is one based on a Bonferroni inequality, namely $\Pr(\Pi A_i) \geqq 1 - \Sigma \Pr(\bar{A}_i)$, which allows the joint confidence level to be bounded in terms of individual confidence levels, independently of the correlations.

Table 615, for use with the Bonferroni solution, gives the $1 - 0{\cdot}05/2k$ point of Student's distribution for $k = 1(1)10$, 15, 20, 50, and for degrees of freedom, $\nu = 5$, 10, 15, 20, 24, 30, 40, 60, 120, ∞. In the three following tables ν is the same, and $k = 1(1)8$. Table 616 gives the 95 per cent. point of the Studentised maximum modulus, is based on work of Pillai & Ramachandran, and applies when variances are assumed equal. Table 619a gives $\{kF_{0.95}(k, \nu)\}^{\frac{1}{2}}$ (F = Snedecor's variance ratio), applies when variances are equal, and is based on Scheffé's simultaneous confidence intervals for linear contrasts. Table 619b gives $\{k\nu F_{0.95}(k, \nu-k+1)/(\nu-k+1)\}^{\frac{1}{2}}$, applies for arbitrary unknown variances, and is based on Hotelling's T.

(R. J. Buehler)

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FIDELIS, E. & ZIELIŃSKI, R. (Math. Inst., Polish Academy of Sciences, Warsaw) **4.3** (3.8)

Estimation of distribution parameters by means of statistics from an ordered sample—*In Polish*

Zast. Mat. (1960) **5**, 149-154

Let $x_1^* \leqq x_2^* \leqq \ldots \leqq x_n^*$ be an ordered sample of size n from a population with a symmetric distribution. If n is even, then denote by x the mean of the values $x_k^*, \ldots, x_{n/2}^*$ and by y the mean of the values $x_{(n/2)+1}^*, \ldots, x_{n-k+1}^*$. In other words, we remove from the ordered sample the $k-1$ largest and the $k-1$ smallest values and then we compute the means of the smaller and larger halves of the remaining part of the sample; k being the parameter of the procedure. Similarly, if n is not even, then x is the mean of the values $x_k^*, \ldots, x_{(n-1)/2}^*$ and y is the mean of the values $x_{[(n-1)/2]+2}^*, \ldots, x_{n-k+1}^*$.

The authors are concerned with using $(x+y)/2$ as the estimate of the mean, and $(y-x)/r_{nk}$ as the estimate of standard deviation in a normal population and they determine the values r_{nk} rendering the last estimate unbiased and give a table for $n = 3(2)9$ and $k = 1(1)4$. They also find the coefficients in a linear form in x and y so as to get unbiased estimators of a and b for a uniform distribution over the segment $a \leqq t \leqq b$.

(S. Zubrzycki)

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HÁJEK, J. (Math. Inst., Czechoslovak Academy of Sciences, Prague) **4.9** (8.2)

Optimum strategy and other problems in probability sampling—*In English*

Čas. pěst. mat. (1959) **84**, 387-424 (13 references)

We have an universe S consisting of arbitrary elements, that is, of numbers 1, ..., N. We select a subset s from S by a random experiment, so that every $s \subset S$ has a known probability $\Pr(s)$. The selected subset will be called a sample, and the system of probabilities $\{\Pr(s); s \subset S\}$ will be called a sampling strategy. Let $y_1, \ldots, y_N$ be some unknown values associated with the elements 1, ..., N, the total $Y = \sum_{i=1}^{N} y_i$ which is to be estimated on the basis of values y_i ascertained on the sample, that is for $i \in s$. An estimator $\hat{Y}$ of the form $\hat{Y} = \sum_{i \in s} y_i w_i(s)$, where $w_i(s)\,(i \in s,\ s \subset S)$ are arbitrary weights, depending not only of i but also of s, is called a linear estimator. The system of weights $\{w_i(s),\ i \in s,\ s \subset S\}$ will be called an estimating strategy. A general estimator is of the form $t = t(s, y)$, where the observation (s, y) involves the knowledge of s and of the values y_i for $i \in s$.

A linear estimate is called representative with respect to values $z_1, \ldots, z_N$ if the equation $\sum_{i \in s} z_i w_i(s) = \sum_{i=1}^{N} z_i$ is fulfilled with probability one. In the paper it is shown that the mean-square deviation $M(\hat{Y} - Y)^2$ for an $\hat{Y}$, which is representative with respect to $z_1, \ldots, z_N$, may be expressed as a weighted sum of the expressions $\left(\frac{y_i}{z_i} - \frac{y_j}{z_j}\right)^2$. A further result concerns a use of sufficient statistics in probability sampling: if s' is a "sample" which involves not only the information as to which elements have been selected but also the order and number of times every element has been selected, and we consider the observation (s', y), then the observation (s, y), where s has been defined above, is a sufficient statistic.

The paper contains a detailed discussion of linear estimates for rejective sampling, consisting of n independent drawings of one element with probabilities $\alpha_1, \ldots, \alpha_N$ provided that results where an element is selected more than once are rejected. A further method of sampling with varying probabilities, called permutation sampling, is presented.

The main results concern optimum sampling and estimating strategy formulated in lines of papers by

continued

HÁJEK, J. (Math. Inst., Czechoslovak Academy of Sciences, Prague) **4.9** (8.2)
Optimum strategy and other problems in probability sampling—*In English* *continued*
Čas. pěst. mat. (1959) **84**, 387-424 (13 references)

Cochran and Godambe. Primarily, it is supposed that $y_1, \ldots, y_N$ are realisations of independent random variables with mean values $My_i = n_i$ and variances $M(y_i - n_i)^2 = d_{ii}$, and costs equal $\sum_{i=1}^{N} c_i \pi_i$, where π_i is the probability of inclusion of the ith element in the sample. It is shown that a strategy is optimum, if the linear estimator is firstly, representative with respect to values $n_1, \ldots, n_N$, secondly, is of the form $\sum_{i \in s} y_i/\pi_i$ and thirdly, the π_i's are proportional to numbers $\sqrt{(d_{ii}/c_i)}$. This result is illustrated by examples and the aspects of its application are discussed.

It is then supposed that $y_1, \ldots, y_N$ is a realisation of a stochastic process with stationary correlation function and with stationary coefficients of correlation, and it is shown that the best strategy is given by systematic sampling with generally unequal probabilities, which generalises and strengthens the well-known result by Cochran.

(J. Hájek)

LATTER, B. D. H. & ROBERTSON, A. (Institute of Animal Genetics, Edinburgh) **4.3** (3.5)
Experimental design in the estimation of heritability by regression methods—*In English*
Biometrics (1960) **16**, 348-353 (2 references, 1 figure)

Broadly speaking, there are two different kinds of technique for the estimation of heritability: analysis of variance and regression coefficients. In some special cases the analysis can be cast in either form. The problems of design in analysis of variance technique have been discussed recently by Robertson [*Biometrics* (1959) **15**, 219-226: abstracted in this journal No. **1**/289, **9.4**].

This paper is concerned with experimental design in regression methods; in particular that of offspring on parent. The authors state that the problem is to obtain the most accurate estimate of the regression coefficient and of the heritability that derives from it for a given expenditure of effort. This effort may be perhaps in time of measurement or in money expended in the rearing of the animals. The assumption made is that if n offspring are to be measured in each family then the expenditure of effort can be considered as proportional to n plus a constant, where the constant is dependent on the particular circumstances. The problem and primary concern then is to find the optimum value of n under different conditions.

The authors begin by giving the sampling variance of the regression coefficient. From this point they discuss the deviation of observed family means about the regression line, breaking it up into two parts. The section closes with two approximations for optimum n when the sampling variance of the regression coefficient has a minimum value.

In the next section the authors discuss regression of offspring on single parent value. Three cases are considered and approximations for n are given in each. The three cases are:

(i) each offspring being related only through the measured parent. The offspring of each measured parent form a half-sib group.

(ii) each measured parent has only one mate, so that the offspring are now full sibs.

continued

LATTER, B. D. H. & ROBERTSON, A. (Institute of Animal Genetics, Edinburgh) **4.3** (3.5)
Experimental design in the estimation of heritability by regression methods—*In English*
continued
Biometrics (1960) **16**, 348-353 (2 references, 1 figure)

(iii) all measured parents have the same mate. The groups of offspring are then full sibs with a half-sib family.

Following these three cases is a situation where the family is a full-sib group. This discussion also gives an approximation for optimum n.

In the closing discussion the authors point out that if there is an environmental cause of similarity among members of a family group, the heritability estimate obtained from analysis of variance will be worthless, but not that obtained by the regression method. In this case the optimum family size will be reduced. In general, the optimum size by the regression method is smaller than that for determination of the heritability from the analysis of variance.

(J. J. Bartko)

JÍLEK, M. & LÍKAR, O. (Central Res. Inst. of Food Industr., and Res. Inst. of Food Industr. Econ., Prague) **4.5** (3.2)

Tolerance limits of the normal distribution with known variance and unknown mean—*In English*

Aust. J. Statist. (1960) **2**, 78-83 (9 references, 3 tables)

This paper is a further development of previous work by the same authors, see " Coefficients for the determination of one-sided tolerance limits of normal distribution " [*Ann. Inst. Statist. Math., Tokyo* (1959) **11**, 45-48: abstracted in this journal No. **1**/401, **4.5**].

If $X_1, \ldots, X_n$ is a random sample of n observations from a normal distribution with mean μ and variance σ^2, it is often necessary to find regions R_A, R_B for which

(i) R_A contains at least a given proportion γ of the distribution with given probability β;

(ii) R_B contains a given proportion γ of the distribution on the average.

The paper gives a survey of solutions for the tolerance limits of these regions when μ is unknown but σ^2 is known.

Two tables are given from which one-sided and two-sided tolerance limits of R_A can be found for given values of γ, β, and n. A third table is provided from which one-sided and two-sided tolerance limits of R_B can be found for given values of γ, and n.

(J. Gani)

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MARSHALL, A. W. (Stanford University, California) **4.5** (10.1)

A one-sided analog of Kolmogoroff's inequality—*In English*

Ann. Math. Statist. (1960) **31**, 483-487 (4 references)

The inequality $\Pr\{X \geqq \varepsilon\} \leqq \mathscr{E}(X^2)/[\varepsilon^2 + \mathscr{E}(X^2)]$ for a random variable with zero mean is well known. The author gives a Kolmogoroff-type generalisation for the situation in which $X_1, \ldots, X_n$ are such that $\mathscr{E}(X_1) = 0$, $\mathscr{E}(X_i \mid X_1, \ldots, X_{i-1}) = 0$, $i = 2, 3, \ldots, n$. Namely: if $\sigma_i^2 = \mathscr{E}(X_i^2)$ and $s_n = \sum_1^n \sigma_i^2$ then

$$\Pr\left\{\max_{i \leqq n} (X_1 + \ldots + X_i) \geqq \varepsilon\right\} \leqq S_n/(\varepsilon^2 + s_n).$$

Further, given a positive ε and numbers $\sigma_1^2, \ldots, \sigma_n^2$ one can construct random variables $X_1, \ldots, X_n$ having the properties of the above mentioned theorem and such that the above inequality becomes an equality.

As another generalisation, at least if $n = 2$, the author shows that if ε_1 and ε_2 are given positive numbers then

$$\Pr\{X_1 \geqq \varepsilon_1 \text{ or } X_1 + X_2 \geqq \varepsilon_2\} \leqq \{\sigma_2^2 + \sigma_1^2(\alpha_2/\alpha_1)^2\}/\{\sigma_2^2 + (\alpha_2^2/\alpha_1)\}$$

where $\alpha_i = \sigma_1^2 + \eta_1 \eta_i$, $i = 1, 2$, and $\eta_1 = \min(\varepsilon_1, \varepsilon_2)$, $\eta_2 = \varepsilon_2$. The variables X_1 and X_2 are assumed to be as above. Again given σ_1^2, σ_2^2, ε_1 and ε_2 the equality in the last expression can always be achieved by a suitable choice of X_1, X_2. Several generalisations of this are given for the case $n = 2$ but with less restriction on X_1, X_2.

The author notes that under the original hypotheses on $X_1, \ldots, X_n$ the variables $\sum_{j=1}^{i} X_j = Y_i (i = 1, \ldots, n)$ form a martingale. Correspondingly he states a version of the first inequality for a separable martingale $\{Y_t;\ t \geqq 0\}$ with $\mathscr{E}(Y_t) = 0$, $\mathscr{E}(Y_t^2) = \sigma^2(t)$: namely for positive ε and τ

$$\Pr\left\{\sup_{t \leqq \tau} Y_t \geqq \varepsilon\right\} \leqq \sigma^2(\tau)/\varepsilon^2 + \sigma^2(\tau).$$

This inequality is sharp if $\sigma^2(\cdot)$ is continuous on the right.

(R. Blumenthal)

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MARSHALL, A. W. & OLKIN, I. (Stanford Univ., and Michigan State Univ.) **4.5** (1.5)

A one-sided inequality of the Tchebysheff type—*In English*

Ann. Math. Statist. (1960) **31**, 488-491 (2 references)

The authors generalise the one-sided Tchebysheff inequality by obtaining an upper bound for $\Pr\{x_1 \geqq 1 \text{ or } \ldots \text{ or } x_k \geqq 1\}$ where the x's are random variables with zero means, $\mathscr{E}x_i^2 = \sigma^2$ and $\mathscr{E}x_i x_j = \sigma^2 \rho \quad (i \neq j)$. The theorem is that if $1-\sigma^2 t > 0$ and $k \geqq \sigma^2(k-1)(1+t)$ then $\Pr\{x_1 \geqq 1 \text{ or } \ldots \text{ or } x_k \geqq 1\}$ is equal to

$$\frac{k\sigma^2\{\sqrt{[1+(k-1)\rho][1+\sigma^2-\sigma^2(k-1)(1-\rho)]}+(k-1)\sqrt{1-\rho}\}^2}{\{k+\sigma^2[1+(k-1)\rho]\}^2}.$$

The situation in which one generalises the two-sided Tchebysheff inequality by considering the absolute values of the x's has been treated by Olkin & Pratt [*Ann. Math. Statist.* (1958) **29**, 226-234]. The argument here relies on some of their results.

(R. Blumenthal)

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NAGAR, A. L. (Econometric Inst., Netherlands School of Economics) **4.8** (6.6)

The bias and moments of the general k-class estimators of the parameters in simultaneous equations—*In English*

Econometrica (1959) **27**, 575-595 (6 references, 1 table)

In this paper the author studies the small sample properties of the so-called general k-class estimators of simultaneous equations. He states that little is known about these properties, the only results being those of the multi-dimensional confidence regions [Anderson, T. W. & Rubin, *Ann. Math. Statist.* (1949) **20**, 46-63] and those by Theil for the coefficients of a structural equation [*Proc. Kon. Ned. Akad. Wetensch.* A (1950) **53**, 1397-1412].

The first section of the paper is devoted to the analysis of the bias, to the order of T^{-1}, T being the number of observations in a column vector; and the moment matrix, to the order of T^{-2}, of the general k-class estimators of an over-identified or just-identified single equation which is part of a system of simultaneous equations.

The second part of this paper states two resultant theorems on bias and on the moment matrix. The author obtains the bias of the two-stage least-squares estimator together with an unbiased estimator, to the order of T^{-1}, as a corollary to the theorem on bias (No. 1). The two corollaries of the theorem (No. 2) on the moment matrix give the moment matrix of the two-stage least-squares estimator and the " best " value of k.

Section three of this paper gives the proofs of the theorems stated in section two and finally, the author gives an example illustrating the selection of the best k-value. A table of alternative point estimates of the coefficients in Klein's model I is given which also shows their estimated finite sample bias and standard errors.

(W. R. Buckland)

NEYMAN, J. & SCOTT, Elizabeth L. (University of California, Berkeley) **4.1** (2.2)

Correction for bias introduced by a transformation of variables—*In English*

Ann. Math. Statist. (1960) **31**, 643-655 (9 references)

This paper is concerned with deducing minimum variance unbiased estimates of the effects of experimental treatments expressed in the original units when the analysis is based on "normally" transformed data. The solution is obtained for a broad category of transforming functions.

Under the basic assumption that the transformation used in the analysis of an experiment is faultless, so that the transformed variables exactly follow normal distributions with some postulated means and with the same unknown variance, an expression is derived for minimum variance unbiased estimates for the treatment effects which is valid for a large class of transforming functions. The resulting expression is complex.

A simpler expression is presented for a sub-class of this class of transformations. Included in this sub-class are the square-root transformation, the logarithmic transformation, the angular transformation and the hyperbolic sine transformation. The estimates and the bias are specifically discussed for these four transformations.

The authors mention that formulas for correcting the bias introduced by the transformation of variables in certain particular cases are contained in some of the references but that these formulas do not agree with theirs.

(J. W. Wilkinson)

2/301

SIMPSON, J. A. & WELCH, B. L. (University of Leeds) **4.5** (2.7)

Tables on the bounds of the probability integral when the first four moments are given—*In English*

Biometrika (1960) **47**, 399-410 (4 references, 1 table, 1 figure)

There is a long history of this problem, given the moments of a distribution to specify as narrowly as possible the probability integral of that distribution. The authors are concerned with the case in which four moments are given. Tchebycheff's inequality is commonly applied to the case where only the mean and standard deviations are available, and the authors note the improvement in the bounds for the probability integral obtained by the introduction of the third and fourth moments. The case where there is some restriction of the range is also discussed.

Using the usual notation, $\Delta = \beta_2 - \beta_1 - 1$ (the boundary of Pearson's impossible area), the table gives the upper and lower bounds to the probability that a random variable is less than a quantity α given Δ, β_1 and β_2. The numerical values considered are $\alpha = |\,10\,|$, (1)...$|\,3\,|$, $|\,2{\cdot}5\,|$ (0·5)...0·0; for $\Delta = 0{\cdot}0$, 0·5, 1(1), ...4, 6, 8, 12, 16, ∞ and $\beta_1 = 0$, 1, 2, 3, 4, 8 and various values of β_2.

(Florence N. David)

2/302

WALLACE, D. L. (University of Chicago) **4.4** (4.1)

Conditional confidence level properties—*In English*

Ann. Math. Statist. (1959) **30**, 864-876 (10 references)

The confidence level (function) of a region-valued estimation procedure is the probability that the region covers the true parameter and will in general depend on the true parameter value. For any subset of the sample space, define a conditional confidence level (function) as the conditional probability, given that the sample lies in the subset, that the region covers the true parameter.

A variety of properties of conditional confidence level behaviour of region-valued estimation procedures are defined. Of these, the most important are the properties $S_2(\alpha)$ and $S_1(\alpha)$. A procedure is $S_2(\alpha)$ if there is no subset for which the conditional confidence level is always (that is for all parameter values) less than α or always greater than α. A procedure is $S_1(\alpha)$ if there is no subset for which the conditional confidence level is always less and bounded away from α. These properties are irrelevant in general, but are desirable properties for a confidence procedure with confidence level α. Fisher's criticism [*J. R. Statist. Soc.* B (1956) **18**, 56-60] of the Welch solution of the Behrens-Fisher problem is equivalent to showing that the level α Welch procedure does not have property $S_1(\alpha)$. Discussion of the desirability of these properties, called by different names, and examples of their violation have been given by Buehler [*Ann. Math. Statist.* (1959) **30**, 845-864]. The present paper obtains and illustrates sufficient conditions for a procedure to have the properties $S_1(\alpha)$ or $S_2(\alpha)$.

A region-valued estimation procedure is said to be " level α Bayes " if it can be generated as follows. For some prior distribution on the parameter space, and for each sample, the posterior distribution is obtained by Bayes theorem. For each sample, a region in the parameter space is chosen having posterior probability α. A procedure is said to be " level α weak Bayes " if it is generated as above, except that in place of a valid prior distribution, a non-negative, but not finitely integrable function is used formally in Bayes theorem, with the requirement that the posterior distribution obtained be, for each sample, a valid distribution.

Principal results are that any " level α Bayes " procedure has property $S_2(\alpha)$ and any " level α weak Bayes " procedure has property $S_1(\alpha)$. Level α Bayes or weak

continued

WALLACE, D. L. (University of Chicago) **4.4** (4.1) *continued*

Conditional confidence level properties—*In English*

Ann. Math. Statist. (1959) **30**, 864-876 (10 references)

Bayes procedures will generally not have constant confidence level α. But if a procedure with confidence level α is simultaneously (say) a level α weak Bayes procedure for some prior " distribution ", then the confidence procedure has conditional behaviour no worse than the $S_1(\alpha)$ property. The weak Bayes result is used to show that the property $S_1(\alpha)$, but not generally $S_2(\alpha)$, holds for the usual normal theory confidence procedures based on t, χ^2 and F, and also the Pitman confidence procedure for location and scale parameters. The Behrens-Fisher fiducial procedure has $S_1(\alpha)$, but the Welch solution does not.

A property weaker than $S_1(\alpha)$ and some associated results are given appropriate for such discrete confidence problems as those for binomial and Poisson parameters.

(D. L. Wallace)

SWAMY, P. S. (University of Poona) **4.3** (2.8)

Estimating the mean and variance of a normal distribution from singly and doubly truncated samples of grouped observations—*In English*

Bull. Calcutta Statist. Ass. (1960) **9**, 145-156 (9 references)

In this paper the author derives the maximum likelihood equations to estimate the mean and variance of a normal distribution from singly and doubly truncated samples of grouped observations with known truncation points, when the number of unmeasured observations is

(i) unknown,

(ii) known separately for each " truncated tail ",

(iii) known jointly for the two " truncated tails ".

These three cases were considered by B. Raja Rao [*Proc. Nat. Inst. Sci. India* A (1958) **24**, 366-376] for discussing the relative efficiencies of best asymptotic normal estimates. The maximum likelihood equations are shown to be easily solvable with the aid of Gjeddebaek's tables of Z_1 and Z_2 functions.

Asymptotic variance and covariance matrix of the estimates is obtained in each of these cases and an investigation is made of the loss of information due to truncation. A numerical example is given illustrating the practical application of the results.

(B. R. Rao)

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BENNETT, B. M. (University of Washington, Seattle) 5.6 (5.1)

Note on the power function of the X_n test in genetics—*In English*

Skand. Aktuartidskr. (1959) **42**, 1-5 (4 references, 1 table)

The event E is supposed to occur k times in a sequence of n independent trials. Let X_n be the rank total

$$X_n = y_1 + 2y_2 + \ldots + ny_n$$

where $y_i = 1$ if E occurs, and $y_i = 0$ if E does not occur on the ith trial. Haldane & Smith proposed X_n as a criterion for testing the randomness of the k occurrences of E, more specifically for testing birth-order effect [see *Ann. Eug.* (1948) **14**, 117-124]. X_n is identical with the statistic in Wilcoxon's test, and related in a simple way to the statistic used in the Mann-Whitney test.

In a previous paper [*Skand. Aktuartidskr.* (1956) **39**, 11-18] the author obtained the conditional distribution of X_n given k as a function of $p_1, p_2, \ldots, p_n$, where p_i is the probability that E occurs in the ith trial. In particular the case

$$\log [p_i/(1-p_i)] = \tau + (i-1)\lambda$$

was considered.

In the present paper the power of the one-sided test (with rejection region $X_n \leqq X_\alpha$) is tabulated for selected values of k, n, α, and λ; the λ-values ranging from $-1{\cdot}0$ to $-0{\cdot}05$.

The author also discusses the modifications which the distribution of X_n undergoes when the number of trials is made dependent on the outcome of the trials. If for example E is the birth of a child affected with some disease, the appearance of one or more E might cause the parents to cease their family. The author gives formulae for the distribution of X_n under the following two rules for terminating the sequence:

(i) Terminate if two successive E's appear,

(ii) Terminate if E occurs in the first trial.

The effect of (i) and (ii) on the distribution is also illustrated by values of the mean and variance of X_n in the case $n = 6$, $k = 2$, $\lambda = 0$.

(B. Matérn)

2/307

BERTRAM, G. (Techn. Hochsch., Hannover) 5.7 (5.3)

Sequential analysis for two binomial experiments—*In German*

Zeit. angew. Math. Mech. (1960) **40**, 185-189 (4 references, 3 tables, 6 figures)

The standard method in sequential analysis is to construct first a sampling plan (a parallel strip in the plane). According to the outcome of experiments, a certain step-function is sequentially constructed until it reaches one of the boundaries.

In the present note the author reverses this procedure. He first draws a step-function according to his observations and with fixed sample size. He then determines a sampling plan in such a way that the step-function ends at one of the boundaries. He finally claims without proving, that the inference which can be drawn by a correct sequential procedure, will also hold in this particular case.

(W. Vogel)

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BHATE, D. H. (Delhi School of Economics) **5.7** (2.5)

A note on the distribution of successive sums of samples from an exponential population—*In English*

Bull. Calcutta Statist. Ass. (1958) **8**, 13-19 (4 references, 3 tables)

In this note the author solves the following problem: let $x_1, x_2, \ldots$ be random observations from an exponential population. Let $X_r = \sum_{i=1}^{r} x_i$: it is required to obtain the distribution of X_r given that $a_i < X_i < b_i$, $i = 1, 2, \ldots, r-1$, a_i and b_i being linear functions of i with equal gradients.

The author then applies the result to compute the probability of arriving at a decision at the rth stage when testing sequentially a simple hypothesis regarding the parameter of an exponential population against a simple alternative. A sequential test of a simple hypothesis regarding the variance of a normal population against a simple alternative is a special case of this problem when the mean of the population is known.

(S. John)

2/309

COX, D. R. (Birkbeck College, London) **5.7** (5.1)

A note on tests of homogeneity applied after sequential sampling—*In English*

J. R. Statist. Soc. B (1960) **22**, 368-371 (5 references, 1 table)

Suppose that in order to examine a parameter θ, observations $(x_1, \ldots, x_n)$ are obtained, the final sample size n being determined by a sequential sampling rule, for example by a likelihood ratio test with fixed limits. It may now be required to use the observations $(x_1, \ldots, x_n)$ for a different purpose, for example to test a subsidiary hypothesis H_s that has tacitly been made in applying the sequential procedure.

In general the use of a standard fixed-sample-size test is not strictly justified because of the dependence of n on the values $(x_1, \ldots, x_n)$. The object of the present paper is to point out that when the subsidiary hypothesis H_s concerns the homogeneity of the observations the standard tests are often justified through an appeal to permutation distributions.

Several examples are given. One concerns the situation where θ is a binomial probability, and the subsidiary hypothesis is that the value of θ for one group of observations is the same as for a second group; the two groups might, for instance, be males and females. The standard test for the difference between two binomial proportions can be used, despite the dependence of n on $(x_1, \ldots, x_n)$, provided that the entry of males and females into the trial is random.

(D. R. Cox)

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DWASS, M. (Northwestern University, Evanston, Illinois) **5.6** (3.7)

The distribution of a generalised D_n^+ statistic—*In English*

Ann. Math. Statist. (1959) **30**, 1024-1028 (6 references)

Let $F_n(x)$ be the empirical cumulative distribution function of n independent random variables, each distributed according to the same continuous cumulative distribution function $F(x)$. The paper is concerned with the distribution of the random variable,

$$D_n^+(\gamma) = \sup_{-\infty<x<\infty} \{F_n(x) - \gamma F(x)\}.$$

Elementary considerations show that this distribution does not depend on the form of the continuous distribution $F(x)$; hence $F(x)$ is taken to be the uniform distribution function for the interval (0, 1).

When $\gamma = 1$, the variable $D_n^+(1)$ is the usual one-sided goodness-of-fit statistic whose asymptotic distribution was first derived by Smirnoff [*Usephi Matem. Nauk*. (1944) **10**, 179-206]. The paper gives two expressions for

$$\Pr(D_n^+(\gamma) < a) = \Pr(F_n(x) \leqq a + \gamma x,\ 0 \leqq x \leqq 1).$$

The first formula for the case $\gamma = 1$ agrees with one obtained by Birnbaum & Tingey [*Ann. Math. Statist.* (1951) **22**, 592-596]. The second formula, the author feels, would involve fewer computations for numerical evaluation. The second formula lends itself to deriving asymptotic results as $n \to \infty$ and yields some previously published facts concerning Poisson processes.

(D. A. S. Fraser)

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FISZ, M. (University of Warsaw, Poland) **5.3** (5.6)

On a result by M. Rosenblatt concerning the Von Mises-Smirnoff test—*In English*

Ann. Math. Statist. (1960) **31**, 427-429 (4 references)

The author points out and corrects an error involving the following integral formula. Let $S_1(t)$ and $S_2(t)$ denote the empirical distribution functions corresponding to two independent random samples x_k $(k = 1, \ldots, n)$ and y_i $(j = 1, \ldots, m)$ from two populations with the same continuous distribution function $F(t)$. It is shown that

$$\int_0^1 [S_1(t) - S_2(t)]^2 d[(nS_1(t) + mS_2(t))/n+m-t] - \int_0^1 [S_2(t)-t]^2 d[S_1(t)-t] - \int_0^1 [S_1(t)-t]^2 d[S_2(t)-t] = 1/6nm.$$

In the original paper, the right-hand-side of the above equation was claimed to be zero. However, the assertion of Rosenblatt's theorem remains true since he was concerned with n, $m \to \infty$. Rosenblatt's original paper " Limit theorems associated with variants of the von Mises statistic " was published in *Ann. Math. Statist.* (1952) **23**, 617-623.

(P. L. Meyer)

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HALPERIN, M. (National Institutes of Health, Bethesda, Maryland) **5.6** (5.3)
Extension of the Wilcoxon-Mann-Whitney test to samples censored at the same fixed point—*In English*
J. Amer. Statist. Ass. (1960) **55**, 125-138 (9 references, 15 tables)

A non-parametric two-sample test for samples censored at the same fixed point is obtained by extension of the Wilcoxon-Mann-Whitney test. The test is proposed for samples of size n on F and size m on G, where F and G are continuous cumulative distributions. The observations on F and G are denoted by x and y respectively and the point of censoring is denoted T, with all x and y in excess of T being censored; there being r_n censored of the former and r_m censored of the latter, or $r = r_n + r_m$ censored in all. The null hypothesis tested is $F(x) = G(x)$, except possibly in the censored portions.

The statistic employed is denoted by U_c and is equivalent to the Wilcoxon-Mann-Whitney U statistic with an adjustment which treats all censored observations as ties. It is shown that under the null hypothesis the conditional distribution of U_c given r is independent of F. This conditional distribution permits the computation of the exact probability distribution of the test statistic.

The author employs this result to compute lower five per cent. and one per cent. points of U_c for m and n less than or equal to eight and for r less than or equal to $m+n-1$. As a consequence of discreteness the significance points are not exact. However, the exact probabilities of the nominal five per cent. and one per cent. points are given parenthetically; and seven tables each are given for these percentage points with tables ordered by n and entered marginally by m and r, the entries being $\overline{U}_c$, the integer value of U_c nearest the appropriate percentage point, and its corresponding exact probability level.

The exact mean and variance of U_c under the null hypothesis are given and it is shown that the standardised

continued

HALPERIN, M. (National Institutes of Health, Bethesda, Maryland) **5.6** (5.3)
Extension of the Wilcoxon-Mann-Whitney test to samples censored at the same fixed point—*In English*
J. Amer. Statist. Ass. (1960) **55**, 125-138 (9 references, 15 tables)
continued

U_c is asymptotically normal with zero mean and unit variance. Thus a large-sample test is available for values of n, m and r outside the tables. A tabular comparison of exact and asymptotic probabilities is given for $n = m = 8$ which suggests that for no more than 75 per cent. censored observations the asymptotic theory is adequate.

The consistency of the U_c test is shown, after the proof given by Mann and Whitney for the U test, with special consideration for the dependence of the $(n+m)$ variables in the censored case.

(S. Krane)

JÍLEK, M. & LÍKAŘ, O. (Cent. Res. Inst. Food Industry and Res. Inst. Food. Industr. Econ., Prague) **5.6** (3.8)

Table of random sample sizes needed for obtaining non-parametric tolerance regions—*In English*

Zastosowania Mat. (1960) **5**, 155-160 (1 table)

A known one-dimensional result of Wilks about order statistics is as follows: the probability P that the interval between the rth smallest and the sth largest value in a random sample of size n taken from a population with continuous distribution covers at least the proportion p of the population distribution is given by

$$P = I_{1-p}(m, n-m+1),$$

where $I_x(a, b)$ is Pearson's incomplete Beta function and $m = r+s$.

This result and its multi-dimensional generalisations, showing the applicability of the above-mentioned relation in the multi-dimensional case as well as in the one-dimensional are the basis for construction of non-parametric tolerance regions.

The authors have computed a table based on $P = I_{1-p}(m, p-m+1)$ which differs from some other known tables in that it determines n, if m, p, and P are given. The table is computed for $m = 1;\ 2(2)10;\ p$ and $P = 0{\cdot}50, 0{\cdot}75, 0{\cdot}90, 0{\cdot}95, 0{\cdot}975, 0{\cdot}99, 0{\cdot}995, 0{\cdot}999$.

(S. Zubrzycki)

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KHATRI, C. G. (University of Baroda, India) **5.4** (2.6)

On testing the equality of parameters in k rectangular populations—*In English*

J. Amer. Statist. Ass. (1960) **55**, 144-147 (4 references, 2 tables)

Rider's statistic for testing equality of ranges of two rectangular populations and Murty's statistic for testing equality of ranges of two rectangular populations when lower limits are zero are extended to the analogous k-sample tests.

The k-sample test statistic in each case is taken as the minimum over all pairs of the corresponding two-sample statistic. Thus, for testing equality of ranges of k rectangular populations the statistic is $u_{\min}$, the minimum of ratios of all pairs of the sample ranges, and for testing equality of ranges of k rectangular populations when the lower limits are zero the statistic is $v_{\min}$, the minimum of ratios of all pairs of sample maximum values.

Sampling distributions of these statistics are obtained under the null hypotheses and the lower five per cent. points of the statistics are tabulated for the case in which the k samples are of the same size, n. For $u_{\min}$ the table has arguments $k = 2(1)5;\ n = 4(1)10(5)20$ and four-decimal entries, except for $k = 5$ in which case the entries are given to three decimals. For $v_{\min}$ the table has arguments $k = 2(1)11;\ n = 1(1)10(5)30, 40, 60, 100, 500, 1000$, and four-decimal entries. An example is given.

(Note: a minor erratum is found in section 3(b) wherein "$v_{\min}$ (or $v_{\max}$)" should be read for "$u_{\min}$ (or $u_{\max}$)" in the first sentence.)

(S. Krane)

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ROY, J. (Indian Statist. Inst., Calcutta) **5.8** (11.1)

Tests for independence and symmetry in multivariate normal populations—*In English*

Sankhyā (1960) **22**, 267-278 (16 references, 3 tables)

Likelihood ratio criteria for a number of hypotheses of interest in multivariate analysis have, under the respective null hypotheses, a tth moment of the form

$$\prod_{1}^{s} [(\tfrac{1}{2}n+c_j)_t]^{-1}[(\tfrac{1}{2}n+b_j)_t]$$

where the b_j and c_j are constants not involving n (the sample size) and satisfying the inequalities $c_j > b_j$, $j = 1, 2, \ldots, s$; $(m)_t = \Gamma(m+t)/\Gamma(m)$. Box [*Biometrika* (1949) **36**, 317-346], Rao [*Bull. Int. Statist. Inst.* (1951) **33**, II 177-180] and Roy [*Bull. Int. Statist. Inst.* (1951) **33**, II 219-230] provide series expansions in powers of $(1/n)$ for distribution functions of such criteria. If we denote the criterion by L and set

$$r_t = \sum_{1}^{s} (c_j^t - b_j^t),\ t = 1, 2; \quad r = 2r_1,\ a = (r_1 - r_2)/r_1,$$

$$N = n - a, \quad X = N \log_e L$$

and

$$Q_k(x) = \int_x^\infty 2^{-\frac{1}{2}k}\{\Gamma(\tfrac{1}{2}k)\}^{-1}u^{\frac{1}{2}k-1}e^{-\frac{1}{2}u}du,$$

$$\Pr(X \geqq x) = Q_r(x) + N^{-2}a_2[Q_{r+4}(x) - Q_r(x)]$$

correct to terms of order $1/n^2$.

The present paper gives tables of r, a and a_2 by the author for the following hypotheses considered in the references given against them:

(1) Specified groups of variables are mutually independent [*Econometrica* (1935) **3**, 309].

(2) The variates are mutually independent and have equal variances [*Ann. Math. Statist.* (1940) **11**, 204].

(3) The variances are equal and the covariances are equal [*Ann. Math. Statist.* (1946) **17**, 257].

(4) The means are equal, the variances are equal and the covariances are equal [*Ann. Math. Statist.* (1946) **17**, 257].

(5) Specified groups of variables are such that within each group the variances are equal and the covariances are equal, and between groups, the covariances are equal for each pair of groups [*Ann. Math. Statist.* (1948) **19**, 447].

(6) Specified groups of variables are such that within each group the means are equal, the variances are equal and the covariances are equal, and

continued

ROY, J. (Indian Statist. Inst., Calcutta) **5.8** (11.1) *continued*

Tests for independence and symmetry in multivariate normal populations—*In English*

Sankhyā (1960) **22**, 267-278 (16 references, 3 tables)

between groups, the covariances are equal for each pair of groups [*Ann. Math. Statist.* (1948) **19**, 447].

(7) Specified groups of variables are such that within each group the variances are equal and the covariances are equal, and between groups, the diagonal covariances are equal and the off-diagonal covariances are equal [*Ann. Math. Statist.* (1948) **19**, 447].

(8) Specified groups of variables are such that within each group, the means are equal, the variances are equal and the covariances are equal, and between groups, the diagonal covariances are equal and the off-diagonal covariances are equal [*Ann. Math. Statist.* (1948) **19**, 447].

(S. John)

SAVAGE, I. R. (Univ. of Minnesota) **5.6** (5.1)

Contributions to the theory of rank order statistics. The one sample case—*In English*

Ann. Math. Statist. (1959) **30**, 1018-1023 (4 references)

The one-sample problem is analysed using techniques developed by the author in earlier papers [*Ann. Math. Statist.* (1956) **27**, 590-615 and (1957) **28**, 968-977]. Tests considered for this problem are distribution-free and are expressed in terms of the statistic $Z = (Z_1, \ldots, Z_N)$ having $Z_i = 1(0)$ according as the ith smallest value in a sample of N is positive (negative). For sampling from a distribution with density $f(x)$ an N-dimensional integral expression is given for $\Pr(Z)$.

Let a 0 in Z' be replaced by a 1 yielding Z or let a 0 and 1 in Z' be interchanged (by moving the 1 to the right) yielding Z. The author gives conditions under which $\Pr(Z) > \Pr(Z')$. These conditions are fulfilled by the normal distribution.

The results can be used for finding and for analysing tests of the hypothesis that the sample is from a distribution symmetrical about zero, against alternatives such as slippage to the right. Under the hypothesis each possibility for Z is equally probable. Under alternatives inequalities such as $\Pr(Z) > \Pr(Z')$ indicate that Z should be a part of the rejection region before Z'. Properties of the Wilcoxon one-sample signed-rank test that are related to these results are considered.

(D. A. S. Fraser)

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SAVAGE, I. R. (University of Minnesota, Minneapolis) **5.6** (3.8)

Contributions to the theory of rank order statistics. Computation Rules for probabilities of rank orders—*In English*

Ann. Math. Statist. (1960) **31**, 519-520 (3 references)

The author obtained two computational rules for evaluating probabilities involving rank orders when the null hypothesis is not the true hypothesis. These results avoid the usually difficult multiple integrations and are obtained for one and two-sample problems. The derived rules are " back-recursive ": for example, in the two-sample problem, probabilities are computed involving samples of size m and n from probabilities involving size $m+1$ and n. These formulae, although of only limited analytical value, may be quite helpful for Monte Carlo sampling.

(P. L. Meyer)

WEIBULL, M. (Inst. Statistics, University of Lund, Sweden) **5.6** (3.6)

Moments of the difference between means in two samples from a finite population: applied in connection with a randomisation test—*In English*

Skand. Aktuartidskr. (1959) **42**, 36-60 (9 references, 4 tables, 2 figures)

Two samples are chosen at random without replacement from a finite population of N individuals. The sizes of the samples are n_1 and n_2, respectively. Thus $n_1+n_2 \leqq N$. The author considers the distribution of

$$u = (m_1-m_2)/s\sqrt{(1/n_1+1/n_2)}$$

where m_1 and m_2 are the observed means of the two samples and s^2 is the variance of the finite population.

The moments and cumulants of u are given in terms of the polykays of the finite population. Explicit expressions are presented for cumulants up to order eight. It is found that u has a symmetrical distribution if the finite population is symmetrical, or if $n_1 = n_2$: the author also studies the asymptotic qualities of the distribution. The applications of these results concern the use of u for testing the significance of the difference m_1-m_2. If $n_1+n_2 = N$, the test is identical with Pitman's randomisation test for two samples.

Three types of population are investigated numerically. In the first example the population values are 1, 2, ..., N; thus u is related to the test-statistic of the Wilcoxon-Mann-Whitney test. In the second example the values are 0, 1, 3, 6, ..., $N(N-1)/2$. For samples from these populations the author compares the exact probabilities of exceeding certain critical values of u with probabilities computed by the normal approximation, and by the Gram/Charlier A-series. The approximation by the A-series with terms up to and including the fourth order is satisfactory for $n_1, n_2 \geqq 5$. The inclusion of terms of order five and six does not result in substantially better approximations. The normal approximation is not satisfactory within the range of n values studied.

The third finite population has k-statistics coinciding with the cumulants of a normal distribution. The probability of exceeding critical values is approximated by the above-mentioned methods. A comparison is made with the corresponding probability in that marginal distribution of u which is obtained when the finite population is considered as a set of independent observations of a normally distributed random variable.

The author provides a table of approximate critical values of u. The table is entered with the coefficients of the terms of order three and four in the Gram/Charlier series.

(B. Matérn)

2/321

WITTING, H. (Univ. of Freiburg and Univ. of California, Berkeley) **5.6** (5.1)

A generalised Pitman efficiency for nonparametric tests—*In English*

Ann. Math. Statist. (1960) **31**, 405-414 (7 references, 4 tables)

To compare two tests of a hypothesis, one may consider the different sample sizes necessary for the different tests to achieve a specified power against a specified alternative. Pitman's criterion of asymptotic relative efficiency, which is at present essentially the only practical criterion available for comparing non-parametric tests is simply the limit of the ratio of the two sample sizes as either the specified power approaches unity, or as the specified alternative " approaches " the null hypothesis. The purpose of this paper is to obtain rates of convergence of this ratio to its limit, namely the asymptotic relative efficiency, thus allowing comparative inferences to be made between tests for smaller sample sizes than was previously possible.

The technique used by the author is the expansion of the power functions of the appropriate tests by means of Edgeworth's series, up to terms of order $O(1/n^2)$, where n is an integer which determines the sample sizes. In a one-sample problem, n may be taken as the sample size itself.

The author works out explicitly the rates of convergence for the following comparisons:

(i) For the two-sample problem, the Wilcoxon test is compared with both the $\bar{x}$ test and the t test for both one-sided and two-sided normal alternatives.

(ii) For the one-sample problem; the sign test is compared with the $\bar{x}$ test and the t-test for one-sided normal alternatives. Some numerical computations are also provided.

(R. Pyke)

ADAMS, E. & MESSICK, S. (Univ. California, Berkeley and Educational Testing Service, Princeton) **6.4** (–.–)

An axiomatic formulation and generalisation of successive intervals scaling—*In English*

Psychometrika (1958) **23**, 355-368 (27 references)

A formal set of axioms is presented for the method of successive intervals and directly testable consequences of the scaling assumptions are derived. Then by a systematic modification of basic axioms the scaling model is generalised to non-normal stimulus distributions of both specified and unspecified form.

Scaled values obtained in this model, at least under certain circumstances, are unique up to a linear transformation and have two interesting consequences for the original successive intervals model based on the normality hypothesis.

(i) If in the errorless case the original model fits, then no other successive intervals model which assumes a different form for the distribution functions will fit. The reason for this is that the forms of the distribution functions are determined by the values of $p_{s,\,i}$ lying above the point t_i. Hence, if the t_i are determined up to linear transformation, so are the curves $p_{s,\,i}$.

(ii) Where the normality assumption does not fit the data it is theoretically possible to use the present generalisation to obtain the scale. Then the deviation in the scale values from those obtained under a normality requirement can be evaluated. This, at least in principle, provides a second kind of goodness-of-fit besides the usual least-squares regression methods employed where the data do not exactly fit the Thurstone model.

The present axiomatic characterisation of a well-established scaling model was attempted because of certain advantages which might accrue: Firstly, an ease of generalisation that follows from a precise knowledge of formal properties, secondly an ease of making comparisons between the properties of different models. An important outcome of their analysis is that the assumption of normality has directly verifiable consequences and should not be characterised as an untestable supposition.

(R. E. Stoltz)

BARBANCHO, A. G. (Instituto Nacional de Estadística, Madrid) **6.6** (–.–)

Identification of complete models—*In Spanish*

Estadística Española (1960) **6**, 5-25 (8 references)

The author considers the problem of identification of relations in complete models; that is the possibility of distinction among relations in determined systems of equations, from the statistician's point of view. He gives special attention to the so-called multi-equation lineal stochastic models. Examples are given to show analytically and graphically whether or not the system is identifiable.

Stating the problem in a general form the author considers models with random disturbances; classified in identifiable, unidentifiable and not exactly identifiable, corresponding to the usual terms applying to structural models: identifiable, underidentifiable, and overidentifiable. Necessary and sufficient conditions for identifying coefficients in structural relations are given; as well as practical rules for deciding if a structural equation is identifiable on the basis of the given system itself.

(F. Azorín)

BROWN, T. M. (Ottawa, Canada) **6.6** (4.8)

Simplified full maximum likelihood and comparative structural estimates—*In English*

Econometrica (1960) **27**, 638-653 (10 references, 2 tables)

In this paper the author, after commenting on the much greater interest now taken both in economic theory and, for economic policy considerations, of macroeconomic models, states that with the present knowledge the full information maximum likelihood estimates with unrestricted correlations among contemporaneous disturbances is the best theoretical method of estimation. It is essentially a large sample method. Even with the simplification afforded by assuming that the disturbances vector has a multivariate normal distribution, the computations are formidable and bulky. Less difficult methods have been developed; for example, the limited information maximum likelihood method by Anderson, T. W. & Rubin, "Estimation of the parameters of a single stochastic difference equation in a complete system" [*Ann. Math. Statist.* (1949) **20**, 46-63] from a suggestion by Girshick.

The main purpose of this paper is to explore a simplification of the mathematics and the computational layout for the full information method. After working through the basic theory of estimating the parameters of a complete economic model by this full information method a simplified and condensed routine is developed. This layout makes for shorter computations because it reduces the direct use of restriction matrices and eliminates the redundant and rather bulky computations involving a kronecker product. The routine is then applied step-by-step to the estimation of the parameters of a known numerical model (Model X) which has generated its own data.

Various other methods of estimation are less costly; viz., least-squares regression, limited information single equations, limited information subsystem and diagonal covariance. Some of these methods are tested both on the above synthetic data and on actual data of a small Canadian model (Model Y) in order to obtain a preliminary assessment of the efficiency of the various methods. Models X and Y are estimated by each method in turn and tables are given to illustrate the results. The author uses the word "efficiency" to mean the inverse function of both bias and computational costs: bias is considered relative to the size of the corresponding parameter. In order "to appraise the relative values of the various methods, computational

continued

BROWN, T. M. (Ottawa, Canada) **6.6** (4.8)
continued

Simplified full maximum likelihood and comparative structural estimates—*In English*

Econometrica (1960) **27**, 638-653 (10 references, 2 tables)

costs are weighted against the estimates of bias, these weights will change with the needs of a particular problem and available resources". In his conclusion the author hopes that continued empirical research will provide even better estimates of the efficiency of the various methods. He says that until more definite results become available, the present results suggest that a good research procedure is to use least-squares or limited information programmes for probing alternative hypotheses and followed by full information maximum likelihood or truncated full maximum likelihood programmes for making final estimations.

(W. R. Buckland)

BASMAN, R. L. (General Electric Corp., Richland, Washington) **6.6** (0.6)

The computations of generalised classical estimates of coefficients in a structural equation—*In English*

Econometrica (1959) **27**, 72-81 (11 references, 7 tables)

This paper illustrates the computation of some generalised classical linear estimates of coefficients in a structural equation. The author states that he has presented his facts in a concise manner suitable for comprehension by the non-statistical reader and that his present paper is a natural sequel to an earlier one, " A generalised classical method of linear estimation of coefficients in a structural equation " [*Econometrica* (1957) **25**, 77-83]. This present paper is also supplementary to the paper by Girshick & Haavelmo, " Statistical analysis of the demand for food: examples of simultaneous estimation of structural equations " [*Econometrica* (1947) **15**, 79-110]. This latter paper deals with the computation of limited information single equation estimators.

For the purpose of comparing the limited information single equation and the generalised classical estimation techniques, where these lead to different results, the author applies the generalised classical method to the three over-identified structural equations of the Girshick Haavelmo model.

The first part of the paper is devoted to an explanation of the composition of the Girshick-Haavelmo model, a table showing the data used in the example and instructions for the preliminary computations. Section four describes the computation of a single structure equation: the author states that he is using the forward Doolittle method [see Klein, *A text book of Econometrics* (1953) pp. 151-165. Evanston: Row, Peterson] in order to carry out the computations.

In conclusion the author points out that the computations which he has outlined constitute the essential steps in the generalised classical estimation procedure and he has used numerical illustrations to illustrate the practical applications of this procedure. He makes some comparisons between the results of limited information single equation and generalised classical estimation procedures and discusses at some length how such comparisons should be made. The need is stressed for further computational research if accurate comparisons are to be made.

(W. R. Buckland)

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CURETON, E. E. (University of Tennessee) **6.9** (6.2)

Note on $\phi/\phi_{\max}$—*In English*

Psychometrika (1959) **24**, 89-91 (2 references)

Using conventional methods for computing the fourfold correlation coefficient, ϕ, the author first points out a serious limitation on the use of that coefficient. Since ϕ is a correlation coefficient, it should be free to vary within the limits of ± 1. These limits, however, can only be reached using the ϕ coefficient when certain conditions are met, that is to say, when marginal proportions are 0·5.

The author proposes a solution to the problem which enables the limits of ± 1 to be always attainable. This is accomplished by dividing ϕ by $\phi_{\max}$, $\phi_{\max}$ being the maximum value of ϕ consistent with given marginal proportions. A formula is given for use when ϕ is positive, and an alternative is given when ϕ is negative. In addition, formulas which allow easier computation and data handling are also given.

(D. Schum)

2/328

DUDZINSKI, M. L. (Division of Math. Statist., CSIRO, Canberra) **6.1** (4.3)

Estimation of regression slope from tail regions with special reference to the volume line—*In English*

Biometrics (1960) **16**, 399-407 (3 references, 4 tables)

This paper is an extension of the approximate method given by M. S. Bartlett [*Biometrics* (1949) **5**, 207-212] for estimating the regression line slope from the join of means of tail regions for evenly spaced values of the predicting variable. The author applies this general study to the fitting of a regression line to commercial volume on sectional area at breast height in stands of even-aged trees. The problem is explained in detail with statements justifying the assumptions made. Array variances were assumed to increase with an increase in the dependent variable. The author proceeds to show that the tail-region method for slope estimation is quite satisfactory under these conditions.

Other investigations are cited in which the tail region method was used, assuming both constant and changing array variances. Tables demonstrating optimum tail region sizes and efficiencies for two β-distributions are given.

The author presents data on measurements from individual trees on three separate plots. The initial task included a least-squares fit to each set of data, omitting the concept of changing array variances: later a close observation was made on the squares of deviates of actual values from the estimates. This was done in order that a general functional relationship could be derived determining array variance as a function of diameter. No such generalisation was found. Following a description of the method for computing weights in order that a weighted regression line can be fitted to the data, tables concerning the goodness of the weight function are given. At this point the slope is estimated, using both weighted and unweighted regression, these results are compared with the less laborious task of estimating by joining tail region means. Tables showing results of calculations for all three tree sites are given, these results including efficiencies of the slope estimators for each of the three methods.

In concluding, the author emphasises the importance of studying the possible heteroscedasticity property in a regression study: if this is not taken into consideration, the result will produce an unbiased but less efficient estimate of the slope.

2/329 (R. H. Myers)

GURLAND, J., LEE, I., & DAHM, P. A. (Iowa State Univ., Ames) **6.8** (4.3)

Polychotomous quantal response in biological assay—*In English*

Biometrics (1960) **16**, 382-398 (12 references, 3 tables)

A quantal response in biological assay has usually two possible outcomes. In a bio-assay, such as that based on mortality of the housefly, there are however, three possible outcomes; namely alive, moribund and dead. More generally in other types of assays, many outcomes might be possible; for example varying degrees of moribundity besides alive and dead. If a response is polychotomous it is more efficient to use this information explicitly in analysing the data rather than pool certain outcomes to make the response dichotomous.

In this paper the authors present general results for a trichotomous response assay, in which the estimates of the parameters are obtained by minimising appropriate chi-square expressions. These chi-square expressions are generalisations of χ^2 (Normits) and χ^2 (Logits) of Berkson [*Biometrika* (1957) **44**, 411-435; *J. Amer. Statist. Ass.* (1955) **50**, 529-550] and have the same asymptotic properties of consistency and efficiency. The results are presented in the context of insecticidal assays based on mortality and moribundity of the housefly. A general procedure is also described for dealing with any number of outcomes; for example, different degrees of moribundity and a general extension of Berkson's minimum Normit and Logit chi-square technique is indicated in an appendix.

The authors also consider the problem of a trichotomous parallel line assay and include as an illustration the analysis of an experiment on Gunthion residue.

(H. O. Posten)

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HOOPER, J. W. (RAND Corp., Santa Monica, California) **6.6** (6.2)

Simultaneous equations and canonical correlation theory—*In English*

Econometrica (1959) **27**, 245-256 (8 references)

In this paper the author has a double purpose; firstly, to contribute towards the understanding of simultaneous equations estimation as a specific part of the more general theory of multivariate analysis, and secondly, he aims to show by analogy to single-equation systems: "in single-equation systems there are regression coefficients which measure systematic relationships between the variable to be explained and the explanatory variables; and there are also correlation coefficients which measure the extent to which these systematic relationships do, in fact, explain the fluctuations in the variable to be explained."

"Regarding simultaneous equations, only the former aspect has been developed and it is the second object of this paper to develop a generalised correlation coefficient which measures the extent to which the systematic relationships explain the fluctuations in the set of all jointly dependent variables."

In the first part of the paper the author uses a topic from multivariate analysis, canonical correlation theory, in the development of a generalised correlation coefficient for simultaneous equation systems. He first presents canonical correlation theory within the framework of simultaneous equations and later the generalised correlation coefficient is developed from these results.

The asymptotic variances of this statistic are derived and some special problems in dealing with particular relation to econometric equation systems are discussed. An example is given in which the results obtained in the previous section of the paper are applied to an econometric model: Tintner's model of the American meat market [*Econometrics* (1952) New York: Wiley].

(W. R. Buckland)

KARON, B. P. & ALEXANDER, I. E. (Princeton University, N.J.) **6.9** (6.5)

A modification of Kendall's tau for measuring association in contingency tables—*In English*

Psychometrika (1958) **23**, 379-383 (3 references)

A coefficient of association τ' is described for a contingency table containing data classified into two sets of ordered categories. Within each of the two sets the number of categories or the number of cases in each category need not be the same. $\tau' = +1$ for perfect positive association and has an expectation of zero for chance association. In many cases τ' also has -1 as a lower limit. The limitations of Kendall's τ_a and τ_b and Stuart's τ_c are discussed, as is the identity of these coefficients to τ' under certain conditions. Computational procedure for τ' is given.

A test of significance for the hypothesis of no association exists, since, for all τ statistics, the test of significance is based on the distribution of S and not of τ.

(R. E. Stoltz)

KASTENBAUM, M. A. (Oak Ridge Nat. Lab., Tennessee) **6.9** (5.2)

A note on the additive partitioning of chi-square in contingency tables—*In English*

Biometrics (1960) **16**, 416-422 (6 references, 2 tables)

This paper is concerned with the partitioning of the asymptotic chi-square statistic involved in the usual test of independence of rows and columns in a $r \times s$ contingency table. The author presents formulae for the partition into one-degree-of-freedom sums of squares representing sets of $(r-1)$ $(s-1)$ orthogonal, interaction-type contrasts on the cells of the table for a broad class of such contrasts.

The author notes that the overall test of independence is equivalent to a test of the hypothesis that s multinomial samples of r-cells have been drawn from the same multinomial population and that in that case one would be interested in partitioning the total statistic into components which would allow for tests on subsets of columns (samples). The formulae are given for the partition into three additive components, one representing non-homogeneity of the first m columns with $(r-1)$ $(m-1)$ degrees of freedom, one representing non-homogeneity of the last $s-m$ columns with $(r-1)$ $(s-m-1)$ degrees of freedom, and one representing the contrast of the first m columns with the last $s-m$ columns with $r-1$ degrees of freedom. The author notes that these three components do not necessarily provide indepedent chi-square tests, since real non-homogeneity as measured by one component may affect the distribution and interpretation of another component.

An example is given of the analysis when the non-homogeneity of row-samples is the variation of interest.

(C. C. Thigpen)

2/333

KLINE, W. E. (Board of Education of Baltimore County, Maryland) **6.3** (–.–)

A synthesis of two factor analyses of intermediate algebra—*In English*

Psychometrika (1959) **24**, 343-359 (14 references, 18 tables)

A battery of 18 tests of intermediate algebra and 20 reference tests was administered to two successive second-year algebra classes. Each battery was separately factor analysed by Thurstone methods. The two analyses were synthesised by the Tucker method. The five congruent factors obtained were identified as: verbal comprehension, deductive reasoning, algebraic manipulative skill, number ability, and adaptability to a new task.

The study raised some interesting questions. First, what serious consideration can be given to the factor loading in a single factor analysis, by the Thurstone multiple-factor technique, if in two studies of almost identical nature only five of the twelve factors obtained were congruent. The author raises the question of whether it would be wise in the future to conduct all factor analysis studies in two parts, to analyse each part separately, and then to test the two parts for congruence.

The author suggests the possibility that Tucker's technique crams a maximum amount of congruence into the first few factors, and thereby creates factors that are psychologically complex and somewhat obscure. He feels that this is added to the difficulty of identifying the factors with any large degree of assurance, and, too, it is added to the resulting high coefficience of correlation between the factors after rotation.

(R. E. Stoltz)

LORD, F. M. (Educational Testing Service, Princeton, N.J.) **6.4** (2.1)

The joint cumulants of true values and errors of measurement—*In English*

Ann. Math. Statist. (1959) **30**, 1000-1004

Suppose U fallible measurements $x_1, \ldots, x_U$ of a characteristic are made on each of a large number of objects. These measurements and the true value $(= \xi)$ are considered random variables with errors, $e_u = x_u - \xi$, $u = 1, \ldots, U$; the errors e and ξ are termed latent variables.

The cumulants of the U's are denoted by $\kappa_{C_1, \ldots, C_U}$ where C_u is a non-negative integer referring to the variable x_u, and the cumulants of the latent variables are $\kappa_{B_0, \ldots, B_U}$ where the subscript B_0 refers to ξ and the others to the U measurement errors.

The article gives explicit expressions for all latent-variable cumulants Κ through the fourth order in relation to the observed-variable cumulants κ for the case $U = 4$. Formulas are derived for determining any Pth order cumulant $\left(P = \sum_{u=0}^{U} B_u\right)$ of the observed measurements as a linear function of the Pth order cumulants of the latent variables. The case where each error of measurement has zero mean and is uncorrelated with every product of the remaining latent variables is also discussed.

(B. M. Bennett)

NADDEO, A. (Faculty of Statistics, University of Rome) **6.2** (9.2)

Analysis of variability of statistical relations—*In Italian*

R.C. Ist. Lombardo Sci. Lett. (1958) **92**, 621-630 (2 references)

The correlation between two normal variables (linearly dependent one on another) is supposed to be under the influence of associated conditions. These can be considered as particular quantitative or qualitative determinations of a third variable or characteristic, or of a combination of variables and/or characteristics associated with the two principal random variables.

Corresponding to each particular concomitant condition, Bernoulli samples are taken from the two-dimensional normal variable; admitting variations in sample size from case to case. In order to ascertain whether the variations of the sample statistics r, with changing conditions, are significant, and to estimate their direction, recourse is made to Fisher's z transformation so as to have an approximately normal sample distribution.

When the concomitant conditions are particular attributes of a qualitative characteristics, comparing different values of z with a Student's t-test one can ascertain possible significant variations in r. When some of those conditions can be considered a particular determination of a discrete variate, the use of the generalised t permits the application of the criteria of analysis of mean values, introduced by Pompilj and to divide the variations of the z's observed in orthogonal comparisons into simple components, each of which shows the influence on correlation owing to elementary variations of the concomitant conditions.

The criteria discussed are illustrated through an application to data by Eden & Fisher [" Studies in Crop Variation, IV," *J. Agric. Sci.* (1927) **17**, 548-562] where the two main normal variables are the weight of the seed and that of the straw of oats sowed in 96 plots in eight randomised blocks, while the concomitant associated conditions are date of fertilising, quality and quantity of fertiliser.

(A. Naddeo)

PAWŁOWSKI, Z. (Central School of Planning and Statistics, Warsaw) **6.1** (4.1)

On the total differential method and its efficiency in the case of linear regression—*In English*

Zastosowania Mat. (1960) **5**, 97-118

The author considers a series of random variables Y_t depending on a discrete time t. Moreover, it is assumed that the expected value of Y_t is a linear function of h real variables $x_1, \ldots, x_h$, whose values at each time moment may be observed. The co-efficients of that linear function are to be estimated. A further assumption is, that the residuals Z_t, that is the differences between Y_t and the linear function in the variables, form a stationary time-series with zero mean and finite variance.

The author constructs the estimators of the co-efficients in question with the aid of least-squares method; but applied to the differences $Y_{t+1} - Y_t$, $Z_{t+1} - Z_t$, and of all the other variables involved, instead of applying it to the raw values of Y_t, Z_t, etc., as is usually done. Hence the term " total differential method ".

After this general introduction it is proved for $h = 1$ that the estimators so obtained are unbiased and, under some additional assumptions, consistent. The efficiency of total differential estimators and of classical ones is compared. The paper ends with a preliminary discussion of the possibility of applying the total differential method in the case of non-linear regression.

(S. Zubrzycki)

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WEINER, I. B. (University of Michigan, Ann Arbor) **6.9** (5.1)

A note on the use of Mood's likelihood ratio tests for item analysis involving 2×2 tables with small samples—*In English*

Psychometrika (1959) **24**, 371-372 (2 references)

Mood's likelihood ratio test is generally considered an unreliable χ^2 approximation in 2×2 contingency tables containing expected cell frequencies less than five. Probability values were computed for 60 such tables as part of an item analysis for two thirty-item alternative forms of a measure. The rank orders of the items, from the best to the worst differentiators as determined separately by Mood's tests and by Fisher's exact test, correlated 0·97 for one form and 0·96 for the other. Thus, although assumptions for its use were not met, Mood's tests give a very good indication of the relative probability values for the items. This may indicate that where ranking is the goal, as in the item analysis described in the article, the χ^2 approximation by Mood's likelihood ratio test is an adequate statistical test, even with small samples.

(R. E. Stoltz)

2/338

DAS, M. N. & KULKARNI, G. A. (Indian Coun. Agric. Res., New Delhi) **7.4** (9.1)

On the accuracy of current approximate variances of treatment differences in randomised block designs with missing observations—*In English*

J. Indian Soc. Agric. Statist. (1958) **10**, 83-89 (3 references)

In this paper the author states the two principal methods for computing the standard errors of estimated treatment differences in randomised blocks with missing observations, and makes certain comparisons.

The two methods stated are, firstly, the multiplication of the plot variance by the sum of the reciprocals of the actual number of replications of the two treatments involved and calculation of the square root: the second method makes use of an " effective number of replications " instead of the actual number of replications.

The rule suggested by Yates [*Emp. J. Exp. Agric.* (1933) **1**, 129] for getting the effective number of replications is the following: subtract from the existing number of replications half the number of those replicates of the other treatment which are in blocks in which the first treatment is not missing. Later, Taylor [*Nature* (1948) **162**, 262] suggested that replacing the factor 1/2 in Yates' method by $1/(k-1)$ where k is the number of treatments, would produce an improvement.

The present paper makes certain numerical comparisons of the values of the standard error given by the various rules with the exact values which are known in the case of comparisons between two treatments only one of which is affected. The author concludes that Yates' rule tends to over-estimate the standard error, whereas the errors computed by using the actual number of replications are generally smaller than the exact values. An average of the values arrived at by these two rules provides a close approximation to the exact value, closer than values given by Taylor's rule.

(S. John)

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FELDT, L. S. (Iowa State University, Ames) **7.6** (9.2)

A comparison of the precision of three experimental designs employing a concomitant variable—*In English*

Psychometrika (1958) **23**, 335-353 (21 references, 6 tables)

Three techniques are commonly employed to capitalise on a concomitant variate and improve the precision of treatment comparisons: firstly, stratification of the experimental samples and use of a factorial design, secondly, analysis of covariance, and thirdly analysis of variance of difference scores. The purpose of this paper is to compare the effectiveness of these alternatives in improving experimental precision, to identify the most precise design and the conditions under which its advantage holds, and to derive, in the case of the factorial approach, recommendations as to the optimal numbers of levels.

Data presented by the author tend to justify the conclusion that the less stringent assumptions of the factorial design more than compensate for the relatively small advantage in precision which may obtain for covariance. This is especially true in educational and psychological research in which the number of degrees of freedom for error are usually quite large and relatively little is known about the form of the relationship between criterion and control measures. The more general applicability of the factorial design becomes apparent from the relationship between the homogeneity of regression from treatment population to treatment population and the phenomena of interaction between treatments and levels in the two-factor design. It is shown that heterogeneity of regression is equivalent to such an interaction, and the presence of either implies the other.

Heterogeneous regression renders the covariance technique, as it is typically applied in educational and psychological research, somewhat invalid. If the usual covariance model is used, the effects would appear to be more serious than those of non-normality and heterogeneity of variance are to an analysis of variance. Further research may indicate that such violations of assumptions is less serious than heterogeneity of variance or non-normality; however, no such conclusion seems warranted at this time.

(R. E. Stoltz)

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GAITO, J. (Air Crew Equipment Lab., Philadelphia) **7.3** (9.1)

The single Latin square design in psychological research—*In English*

Psychometrika (1958) **23**, 369-378 (10 references, 4 tables)

It has been assumed that interactions must be non-existant for results to be adequate in using the Latin square designs. Gourlay, using a variance component analysis, indicates that for a valid application of the Latin square techniques, however, interactions do not always have to be zero. Furthermore, contrary to McNemar's assertion, he found that too few significant F values might result; a negative F-test bias. Gourlay investigated this problem with reference to two main types of interactions that occur in psychology. Firstly, where each individual or unit receives only one of several treatments and is represented by one measurement in the data. In this case interaction is between main effects. Secondly, repeated measurements are made on the same individuals or groups. In this case earlier measurements may interact with those that follow. The author feels that a more general and instructive procedure would be to determine the components of variance included with each mean-square under four conditions: zero, one, two, and three random variates. The first condition corresponds to the fixed variate model, the last to the random variate model, and the others to the mixed model. He states that this procedure would exhaust all possibilities in the single Latin square design and would permit an evaluation of the behaviour of each test of significance.

The author demonstrates the possible defects inherent in the single Latin square tests of significance when interactions occur under the four conditions mentioned above. The data presented show that as the number of random variates increases the number of unbiased F-tests increases likewise until in the three random variates model all tests are free of bias. Paralleling this trend is an increasing number of valid F-tests as well. In the first two models negative bias occurs most frequently but small positive bias is possible when the triple interaction is present. In the third model all biases are negative. As the number of random variates increases, more tests are insensitive to deviation from the assumptions that interactions must be zero. Less bias will occur with interactions present inasmuch as some tests will be unbiased; however, a fewer number will be valid. Contrary to McNemar's assertions, and in agreement with Gourlay, most of the tests have negative bias: the probability of a type 2 error increases with these Latin square designs for some tests of significance.

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(R. E. Stoltz)

HAYMAN, B. I. (Appl. Math. Lab., DSIR, Christchurch, N.Z.) **7.2** (4.2)

Maximum likelihood estimation of genetic components of variation—*In English*

Biometrics (1960) **16**, 369-381 (6 references, 4 tables)

In this paper the methods of least-squares, weighted-least-squares, maximum likelihood, and one due to Nelder [*Heredity* (1953) 7, 111-119] are applied to the problem of estimating the additive, dominance, and environmental effects in a genetic population. The ordinary least-squares method is unsatisfactory when the observed components of variance are correlated or have unequal errors, and for this reason Mather [*Biometrical Genetics* (1949). London: Methuen] considered weighted-least-squares estimation using weights obtained from replications within the experiment. Assuming normality, the maximum likelihood estimates can be obtained from the weighted-least-squares by an iterative procedure.

The similarities and differences of the estimates are illustrated from results of a breeding experiment on the plant *Nicotiana rustica*, for which four generations were available in 1954 and five in the succeeding year. The analyses are carried through on plant height for 1954, and time of opening of first flower for both years. For the maximum likelihood method, five iterations are calculated although the chi-square goodness of fit stabilises at the second iteration.

In the analysis on height, the methods agree that heritability is high due to additivity of gene action, but that dominance and epistasis are unimportant. However, they differ in assessing the significance of environmental effects.

For the analysis of flowering time, four models are considered ranging from fitting 10 parameters allowing for seasonal changes in both genetic and environmental components, to a model with 4 components not allowing for seasonal changes. It is found that a model having varying environmental but constant genetic components (6 in all) fits the data almost as well as the most general model. A more detailed investigation shows that the environmental effects decrease by about one-half over the two years, but also suggests that the (significant) dominance component has decreased somewhat. These conclusions are based on the maximum likelihood method; if unweighted-least-squares are used the conclusion concerning dominance is that it is not even significant in either year, a contradiction due to the incorrect weighting of the latter method.

(G. A. Watterson)

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LORD, F. M. (Educational Testing Service, Princeton, N.J.) **7.6** (7.5)

Large-sample covariance analysis when the control variable is fallible—*In English*

J. Amer. Statist. Ass. (1960) **55**, 307-321 (9 references, 2 figures, 1 table)

When the control variable, or covariate is subject to errors of measurement, the usual analysis of covariance is no longer valid; its use may lead to equivocal conclusions. In particular, it may point to significant differences among adjusted-group-means when none in fact exist; it may likewise fail to detect such differences when they do exist.

The purpose of this paper is to outline an appropriate procedure for dealing with analysis-of-covariance data in the simple case where there are two groups, composed of individuals on each of whom we have one measurement on the criterion variable and two duplicate measurements on a control variable. The hypothesis to be tested is that the two regressions are collinear.

The test developed in this paper is obtained by a large-sample approach. The basic step is the construction of an asymptotically unbiased estimator of the difference between the adjusted group means. Since the estimator is a continuous function of moments with continuous first derivatives, its distribution is approximately normal for large samples. Once the large-sample standard error of the estimator has been obtained, it is possible to make use of familiar procedures for testing significance.

The author provides an example; not only as a guide for practical computation, but also to illustrate the fact that the method explained can demonstrate a significant deviation from the null-hypothesis in data where no such deviation would be shown by the usual analysis of covariance methods.

(M. Dorff)

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ODEH, R. E. & OLDS, E. G. (Carnegie Institute of Technology, Pittsburg), **7.5** (2.2)

Notes on the analysis of variance of logarithms of variances—*In English*

WADC Technical Note 59-82 (1959), Aero. Res. Labs., Wright-Patterson Air Force Base. iv+47 pp. (15 references, 13 tables)

One of the assumptions needed in the analysis of variance is equality of within-cell variances. Often the validity of this assumption is investigated by the use of Bartlett's modification of the Neyman-Pearson L_1 test. The present report is concerned with the investigation of an alternative test, the analysis of variance of logarithms of variances, which seems to have some more desirable properties.

The technique consists of dividing the observations in each group into sub-groups, applying the logarithmic transformation to the sub-group variances, and then performing an analysis of variance on the logarithms. Let us suppose that we are given random samples of equal size J, one from each of K normal populations. Let y_{kj} be the jth observation in the kth group, $j = 1, 2, \ldots, J$; $k = 1, 2, \ldots, K$, where the y_{kj} are $NID(\mu_k, \sigma_k^2)$. We wish to test the hypothesis $H_0 : \sigma_1^2 = \sigma_2^2 = \ldots = \sigma_K^2$. We divide the observations in each group into M sub-groups each of size A, ($MA = J$, $M>1$, $J>1$), then compute the logarithm of the variance of the observations within each sub-group, and perform a simple one-way analysis of variance on the logarithms of variances. The ratio of the mean-square between groups to the mean-square within groups is denoted by F_L.

It is shown that, under the null hypothesis, F_L has asymptotically, as A increases, an F distribution with $(K-1)$ and $K(M-1)$ degrees of freedom. Since the exact distribution of F_L for small samples is mathematical intractable, a study is made of the Type I error rates if the percentage points of the F distribution are used in testing the significance of F_L. For $K = 2(1)7$, 9, 11; $M = 2(2)8$, 12; and $A = 2(2)8$, 12 it is found that $\Pr[F_L \geqq F_{0\cdot05}]$ is in the range 0·041 to 0·056 and that $\Pr[F_L \geqq F_{0\cdot01}]$ is in the range 0·007 to 0·015, so that no serious errors result from the use of the percentage points of the F distribution. An example of the use of the proposed test is given.

A test in this form would be expected to be " robust " (relatively insensitive to non-normality), which the Bartlett test is not, and would also have the advantage that we could break out single-degree-of-freedom contrasts. The question of the power of the test is now being investigated, and results will be given in a subsequent report.

(H. L. Harter)

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OLDS, E. G. & LEWIS, J. S. (Carnegie Institute of Technology, Pittsburg) **7.5** (2.2)
Notes on the use of F in the analysis of variance of attributes data and on the transformation of measurements data to attributes data—*In English*
ARL Technical Note 60-118 (1960), Aero. Res. Labs., Wright-Patterson Air Force Base. iv+42 pp. (11 references, 14 tables)

In the usual procedures for the analysis of variance, significance tests are based on the assumption that the data come from populations having normal distributions. The recognition that, in practical situations, this assumption is at best only approximately fulfilled has led to a continuing investigation of how lack of normality affects the results of the significance tests. Box [*Biometrika* (1953) **40**, 318-335] and others have shown that the usual F tests on means are " robust ", that is that they are insensitive to lack of normality. The analysis of variance technique therefore remains a potent analytical tool even though a basic assumption inherent in it is not strictly satisfied. An applied statistician, when faced with analysing data that, obviously are not from a normal population, may be able to overlook this fact and still expect to get useful results. Cochran [*Biometrika* (1950) **37**, 256-266] has suggested that the F test might serve as an approximation even when the table consists of 0's and 1's, since both the F test and the conventional χ^2 test assume normality, and it is not obvious which is more sensitive to the assumption. Batson [*National Convention Transactions* (1956) Amerian Society for Quality Control, 9-23] has described a development of the χ^2 test as applied to the analysis of attributes data, and has ascribed to Brandt the formulation of this method, called " factorial chi-square ". This report is concerned with the comparison of this method to the usual analysis of variance as applied to attributes data. In the simple case of n replications at each of k levels of a single factor, both test statistics, called F^* and χ^*, have as their numerator the mean-square between levels. F^* has as its denominator the mean-square within levels, and F^* is treated as an F variable with $(k-1)$ and $k(n-1)$ degrees of freedom. χ^* has as its denominator $\bar{p}\bar{q}$, where $\bar{p}$ is the proportion of zeros and $q = 1-\bar{p}$ is the proportion of ones in the experiment, and χ^* is treated as a χ^2 *degrees of freedom* variable with $(k-1)$ degrees of freedom. Asymptotically

continued

OLDS, E. G. & LEWIS, J. S. (Carnegie Institute of Technology, Pittsburg) **7.5** (2.2)
continued
Notes on the use of F in the analysis of variance of attributes data and on the transformation of measurements data to attributes data—*In English*
ARL Technical Note 60-118 (1960), Aero. Res. Labs., Wright-Patterson Air Force Base. iv+42 pp. (11 references, 14 tables)

(as $n\to\infty$) the distributions of F^* and χ^* are identical, since both converge in distribution to $\chi^2/(k-1)$. On the basis of a complete enumeration of the possible values of F^* and χ^* for several pairs of values of n and k (nk small), it is conjectured that F^* will always be as good as, or better than χ^* if nominal 95 per cent. points are to be utilised.

The problem of finding the best method of transforming measurements data from a two-factor experiment to attributes data is discussed and exemplified.

(H. L. Harter)

RAO, M. M. (Carnegie Institute of Technology, Pittsburg) **7.5** (2.2)

Some asymptotic results on transformations in the analysis of variance—*In English*

ARL Technical Note 60-126 (1960), Aero. Res. Labs., Wright-Patterson Air Force Base. iv+20 pp. (13 references)

Several transformations have been used in the past to stabilise the variance in particular problems of the analysis of variance. Most of these studies use only approximations. The difficulty, or impossibility, of formalising the heuristic argument involved here has been demonstrated by Curtiss [*Ann. Math. Statist.* (1943) **14**, 107-122], who sought to obtain satisfactory asymptotic solutions. Recently, for the logarithmic transformation, Severo & Olds [*Ann. Math. Statist.* (1956) **27**, 670-686] have shown that, if the exact solution is considered, different tests applied to the same data for the same hypotheses have very distinct power properties while all those tests have identical power functions asymptotically. Thus the solutions in the study of transformations can be broadly classified into two groups. The first involves obtaining certain approximate solutions, and the second proving certain asymptotic results. The present note belongs to the latter group.

The problems considered, and the results obtained, in this note may be stated as follows. The asymptotic properties of the power function obtained by Severo & Olds are shown to hold, under similar conditions, also for the square-root transformation (variance assumed known), so that this type of result is not the property of a very special distribution. Clearly, there is no reason to expect the same behaviour of different tests for any such (stabilising) transformation in the non-asymptotic, or exact case.

Next, the problem of testing the mean of a log-normal distribution when the variance is not known is considered. In this case a necessary and sufficient condition for the usual normal-theory analysis of variance procedures to hold asymptotically, as the mean tends to infinity, is that the coefficient of variation (of the log-normal variate) should tend to zero.

Finally certain general results, in the spirit of Curtiss' paper, are presented. These deal with the problems of normal convergence of certain random functions. Sufficient conditions for the applicability of the normal theory are presented for a class of distributions depending on a finite set of parameters with one parameter large, while the others, if any, are relatively small, or are confined to a fixed bounded set in the parameter space. The last section includes some remarks on related problems.

(H. L. Harter)

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STEEL, R. G. D. (Math. Res. Center, Univ. Wisconsin; and N. Carolina State Coll.) **7.7** (5.6)

A multiple comparison sign test: treatments versus control—*In English*

J. Amer. Statist. Ass. (1959) **54**, 767-775 (3 references, 4 tables)

A number of treatments are to be compared with a control. The hypothesis to be tested is that the median of each of the other k treatments equals the median of the control. The test statistic is the vector $(r_1, \ldots, r_k)$, where r_i is the number of times the sign of the difference, ith treatment effect less the control effect, is negative in n trials. (It is assumed that the ith treatment effect and the control effect are never equal. A simple rule of thumb is suggested for use if they are.) If the r_i denoted positive signs, the distribution would remain unchanged. The author demonstrates the combinatorial method for working out the distribution of $(r_1, \ldots, r_k)$ for the case where the null-hypothesis is true. From this k-variate distribution one can then obtain the distribution of r = minimum r_i. This is tabled for $k = 2$, $n = 4(1)10$ and for $k = 3$, $n = 4(1)7$.

If the alternative hypothesis is that at least one of the other treatment medians is greater than the control median, then one rejects the null-hypothesis when r is less than or equal to the tabular one-tail critical value of r. This is tabled for $k = 2(1)9$, $n = 5(1)20$, and probability in the tail of 0·05, 0·01. The ith treatment is declared to have a median significantly higher than the control median, if r_i is less than or equal to the one-tail tabular r.

If the alternative hypothesis is that at least one of the other treatment medians is unequal to the control median, then one should redefine r_i to correspond to the less frequent sign. One then rejects the null-hypothesis when r is less than or equal to the tabular two-tail critical value of r. This is tabled for $k = 2(1)9$, $n = 6(1)20$, and probability in the two tails of 0·05, 0·01. The ith treatment is declared to have a median significantly different from the control median, if r_i is less than or equal to the two-tail tabular r.

In the last section the author conjectures that the distribution of $(r_1, \ldots, r_k)$ is asymptotically multivariate normal and that $(\min r - \mu_r)/\sigma_r$ is distributed approximately as Dunnett's t with infinitely many degrees of freedom. A comparison with their tables gives close agreement. The final approximation used is $r = \frac{1}{2}(n-1-tn^{\frac{1}{2}})$, (Dunnett's t with degrees of freedom equal to infinity).

(S. W. Nash)

ZYSKIND, G. & KEMPTHORNE, O. (Iowa State University, Ames) **7.5** (9.4)

Treatment errors in comparative experiments—*In English*

WADC Technical Note 59-19 (1960), Aero. Res. Labs., Wright-Patterson Air Force Base. iv+78 pp. (45 references, 6 tables, 3 figures)

The primary concern of the present report is with the investigation of a representative number of alternative experimental schemes in which explicit allowance is made for the fact that application of intended amounts of treatment may be subject to error. These representative experimental schemes are, firstly, the completely randomised one-factor experiment; secondly, designs in generalised randomised blocks with three alternative randomisation schemes for assigning treatment attempts to the plots within blocks; and lastly, two-factor completely randomised designs with two alternative randomisation schemes for assigning treatment attempts to experimental units.

In Chapter Two, the authors demonstrate how the question of treatments subject to error may be structurally incorporated into a broader formulation of experimental design. This more general formulation involves notions of the explicit act of randomisation, the use of design and sampling random variables, balance (in the occurrence of levels of a treatment in the experimental design), Σ expansions of the variances of effects, rightmost bracket subscripts, and complete ambivalence. The last three are new notions which are defined in the report. Using these notions, the authors state, without proof a theorem expressing the expected mean-square of a partial sample-mean as a linear combination of Sigma expansions. For further details of this more general formulation, reference is made to *Error Structures in Experimental Designs* [Zyskind (1958) unpublished Ph.D. thesis, Iowa State Univ. Library, Ames]. As a result of this theorem, when certain sets of structural conditions are satisfied, the expected values of mean-squares in the analysis of variance tables have simple and easily specifiable forms. These conditions have been verified to hold in all of the experimental schemes discussed in the present report.

In discussion of these experimental schemes the following procedure is generally used. The population structure is symbolically represented and the population identity is given. Applying their general definitions,

continued

the Σ expansions are explicitly written down. The design and sample random variables are introduced and the statistical model then presented. Finally an analysis of variance table showing the sources of variation and expected values of their corresponding mean squares is shown.

Consideration of the above-described experimental schemes which were investigated allow one to abstract the following overall trends in the conclusions. If treatment errors are operative then repeated independent attempts at realising the preassigned amounts of treatments are to be desired since without them partial or total confounding of treatment amounts and treatment errors will result. The presence of treatment errors tends to decrease the sensitivity of tests of significance for treatment effects. As is also the case with interactions involving experimental units, the presence of interactions involving treatment errors tends not only to decrease further the sensitivity of tests but also to introduce negative bias both into tests of significance for treatments and into the estimation of the treatment component of variation; but the presence of these interactions tends to over-estimate the average variance of estimates of treatment differences. Since the magnitudes of non-additivities are scale-dependent, the results of the present report lend further support to the importance of seeking a scale which as nearly as possible achieves additivity.

(Mary D. Lum)

BHATTACHARYA, A. K. (Indian Institute of Technology, Kharagpur) **8.9** (5.7)

A note on the efficiency of sequential sampling plans based on gauging—*In English*

Bull. Calcutta Statist. Ass. (1960) **9**, 117-121 (3 references)

Sampling plans based on gauging are operationally simpler, though less efficient, compared to those based on variate values. The author demonstrates that in tests of hypotheses specifying the mean of a normal population with known variance, against one-sided alternatives, the loss of efficiency for sequential plans based on gauging is not high.

Observations are drawn one by one and, at each stage the total number of observations having values less than G_1, the number lying between G_1 and G_2 and the number above G_2, are noted. The observations are supposed to come from a normal population with mean θ and variance unity. The hypothesis to be tested is that $\theta = \theta_0$ against the alternative that $\theta = \theta_1 > \theta_0$. The sequential probability ratio test for testing this hypothesis is constructed.

The average sample number varies with the values of G_1 and G_2. By numerical computations the author finds that the average sample number can be minimised by setting $G_1 = \theta_0 - 0{\cdot}61$ and $G_2 = \theta_0 + 0{\cdot}61$. It is noted by the author that the optimum gauge setting is independent of the value of θ_i.

It is of interest to inquire how efficient this procedure is when compared to sampling plans based on variate values. Taking the ratio of the average sample numbers in the two cases as a measure of relative efficiency, the author evaluates the efficiency of the sampling procedure under consideration for $\theta_1 - \theta_0 = 0{\cdot}25$ $(0{\cdot}25)$ $1{\cdot}0$ and $\theta_1 - \theta_0 = 0{\cdot}1$. In each case the efficiency is about eighty per cent., though there is a tendency for it to decrease as $\theta_1 - \theta_0$ increases.

(S. John)

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BRYANT, E. C., HARTLEY, H. O. & JESSEN, R. J. (Univ. Wyoming, Iowa State Coll. and General Analysis Corp.) **8.1** (4.9)

Design and estimation in two-way stratification—*In English*

J. Amer. Statist. Ass. (1960) **55**, 105-124 (7 tables, 11 references)

One may encounter populations for which there are two effective stratifying criteria, both of which are desirable in a sample design. However, the number of permitted observations may be less than the number of strata formed by the usual double stratification technique. The authors present a method which will permit estimation in these cases. Both biased and unbiased estimators are considered.

It is shown that, if the stratification effects are additive in the analysis-of-variance sense, the method is particularly effective. It is stated that if the substrata sizes (formed from the two-way classification of the population according to the stratifying criteria) are proportional to the product of the corresponding one-way strata sizes, the possible loss in efficiency compared to single stratification is trivial. Even in populations in which substrata disproportion is great, one can still use the method effectively by employing a method of allocating, with certainty, some of the sample observations to the substrata.

Variances of both biased and unbiased estimators are given, along with a method for obtaining essentially unbiased estimates of the variances.

(H. O. Hartley)

CURNOW, R. N. (Agric. Res. Council, Statist. Unit, Univ. Aberdeen) **8.6** (8.0)

Consequences of errors of measurement for selection from certain non-normal distributions—*In English*

Bull. Int. Statist. Inst. (1960) **37**, III 291-308 (5 references). Discussion abstracted No. **2**/354

The mathematical problem can be briefly summarised as follows: the true values of a certain characteristic are distributed over an infinite population: observed values of this characteristic are subject to an additive and normally distributed error. A new population is formed by selecting a predetermined fraction of the original population possessing the highest observed values. The distribution of true values within this selected fraction is to be studied.

This type of selection problem occurs frequently in plant breeding work. Assuming that the true values are initially normally distributed and given certain details of the amount of land available and the cost involved, optimal procedures have been proposed for the conduct of plant breeding programmes. The sensitivity of these optimal procedures to departures from normality clearly needs investigating.

Finney, D. J. has provided a method for studying the effects of errors on selection from a general distribution specified by its cumulants. See " The consequence of selection for a variate subject to errors of measurement" [*Rev. Int. Statist. Inst.* (1956) **27**, 22-29]. However, more direct methods are available when the distribution takes one of a certain number of forms. This paper presents the methods involving Hermitian probability functions and recurrence relationships that are being used to study selection with error from rectangular, gamma (or χ^2) and χ-distributions.

(R. N. Curnow)

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CURNOW, R. N. (Report of discussion on a paper by) **8.6** (8.0)

Consequences of errors of measurement for selection from certain non-normal distributions—*In English*

Bull. Int. Statist. Inst. (1960) **37, I** 59-61

Those taking part in the discussion included Hemelrijk, Neyman and Hamaker.

Hemelrijk commented that the proportion P was introduced in the paper as the proportion of varieties selected in the sample; but in the mathematical treatment the quantity η, related to P by $\Pr\{y>\eta\} = P$ was treated as constant. This meant that P was a proportion in the population instead of the sample, thus, selection in reality referred to y, taking a value greater than η, and was only asymptotically equivalent to selecting the fraction P of the largest value in the sample. For large samples there would not be much difference in practice, but, from a theoretical point of view he would like to known whether the relation between P in the sample and η was an asymptotic one. The author replied that the results were asymptotic in that they depended on the assumption that the number of varieties was very large. He hoped to be able to investigate the importance of the assumption.

Neyman stated that both Mrs P. C. Tang and Nash had investigated exactly the same problems: that is, those dealing with the selection of the number of varieties and intensity selection. The author replied that he had referred to Mrs Tang's work but that she had dealt with slightly different problems.

Hamaker, while drawing attention to some industrial applications of the problem hoped that the author would pay some attention to these partially solved applications. The author replied that while certain industrial problems were probably closely analogous to those discussed the work done by O. L. Davies [" The design of screening tests in the pharmaceutical industry ", *Bull. Int. Statist. Inst.* (1959) **36**, III 226-241: abstracted in this journal No. **2**/149 and No. **2**/151 **9.4** and " Some statistical aspects of the economics of analytical testing ", *Technometrics* (1959) **1**, 49-62] was worth mentioning as being probably of rather more direct application to the industrial problems.

(W. R. Buckland)

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HAMAKER, H. C. (Philips Research Labs., Eindhoven) **8.8** (–.–)

Attribute sampling in operation—*In English*

Bull. Int. Statist. Inst. (1960) **37**, II 265-281 (9 references, 4 tables, 4 figures)

Discussion abstracted No. **2**/356

This paper brings a critical survey of attribute sampling techniques as applied in industry to-day. In setting up a sampling table we always attempt a compromise between a number of conflicting requirements. The choice of the most suitable sampling method depends on the features that seem most desirable. Opinions differ and this explains the variety in sampling tables that have been proposed. Various features of these tables are discussed in detail such as the acceptable quality level concept, the relation between lot size and sample size, the use and utility of tightened and reduced inspection, etc.

It is emphasised that the purpose of sampling inspection is not only to separate bad lots from good, but also to control the quality of the lots by making use of the information provided by the samples. This second purpose is too often disregarded. For purposes of control it would be advantageous to use a constant sample size independent of the lot size. This is illustrated by an example.

Finally, the author discusses the question of choosing the most desirable features of a universal sampling standard.

It is pointed out that normal, tightened, and reduced inspection might well be incorporated in a single table provided with sliding scales for both lot size and acceptable quality level. A modification of the well-known Military Standard 105A in this sense is presented in detail.

(H. C. Hamaker)

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HAMAKER, H. C. (Report of discussion on a paper by) **8.8** (–.–)

Attribute sampling in operation—*In English*

Bull. Int. Statist. Inst. (1960) **37**, I 134-138

Those taking part in the discussion were Hald, Barnard, Riordan, Anson, Madeira and Wallis: the author replied.

Hald took the view that, although MIL-STD 105A could be regarded as a convenient catalogue of sampling plans with corresponding operating characteristic curves, as could the author's system be regarded as a collection of single sampling plans, it was not a good thing to throw back to the user the difficult problem mentioned in the first section of the paper. This was, however, the effect of the author's definition of a sampling plan as being merely the sample size and the acceptance number. Instead of a move to greater simplification of plans it was probably better in the long run to try and make them more realistic: for example, with respect to the prior distribution, the costs and also the mechanism of statistical feed-back.

Barnard accepted the economic approach proposed by Hald and gave a practical plan for using a feed-back system to determine whether too much sampling was being carried out. Riordan drew attention to the distinction between factory data and data from performance in service. The latter was both difficult to collect and to process. A difficulty of using MIL-STD 105A was the changing inspection work-loads.

Anson preferred to regard the problem as consisting of two parts (i) the discrimination between good and bad lots; which was a single-stage decision problem when all the information had to be collected in a first sample, and (ii) the control quality of lots; which was a multi-stage decision problem.

The author replied at some length on the difficulty of pursuing the economic viewpoint in practice and gave an example illustrating the difficulty of dealing with data from performance in service.

(W. R. Buckland)

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HAMAKER, H. C. (Phillips Research Labs., Eindhoven) **8.1** (-.-)

Qualitative control of the sample—*In French*

Rev. Statist. Appl. (1960) **8**, 5-40 (24 references, 6 tables, 9 figures)

In this paper the author first states the fundamental principles which are the basis for quality control by sampling and then discusses efficiency (operating characteristic) curves. Graphs are given showing comparisons of several efficiency curves in simple sampling plans.

Quality control as applied to double, multiple and sequential plans which allow economy of observations is also discussed and some comments made on the methods of choosing sampling plans for particular situations.

Acceptable Quality Level is also mentioned as another factor which influences the choice. A figure in this section illustrates the efficiency curves corresponding to the simple sampling plan $P = 0{\cdot}95$ for $p = 2{\cdot}5$ per cent. when the acceptance numbers are 0, 1, 2 and 3.

It is stated that in the well-known Military Standard 105A [*Sampling procedures and tables for inspection by attributes*, U. S. Government Printing Office (1950)] the probability of acceptance corresponding to the Acceptable Quality Level covers the range of numbers as far as 85 per cent. Also included in this section are a study of the important differences in the main objectives of sampling; economic theories which are related to the choice of a sample design and a discussion on the distributions of the percentage of defective lots in the controlled batch. It is stated that there may be no precise information on this distribution: figure four illustrates various forms that the distribution can take. The following paragraphs contain detailed studies with illustrative tables of the relationship between the size of a batch and the size of a sample.

In the latter part of this paper, the special characteristics of the principle sampling plans are examined:

(i) US Military Standard 105A
(ii) Columbia tables
(iii) Philips SSS
(iv) German AWF
(v) German VG 95083
(vi) Swedish military standard
(vii) Dodge & Romig tables.

Special emphasis is placed on the interest to be found in the use of a sample of constant size independent of the batch size. The author investigates the possibility for an international standard of sampling instead of the many plans now in existence.

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(Mlle. Turlot)

KETTMAN, G. **8.8** (8.3)

Comparison of the principal acceptance sampling systems with a new German system—*In French*

Rev. Statist. Appl. (1959) **7**, 5-25 (6 references, 9 tables, 6 figures)

A new sampling system to be called ASQ-UA6 has been proposed by the Sampling Plans Sub-Committee of the German Association for Statistical Quality Control. The author of this paper has discussed this system and also compared it critically with the best known of other sampling systems; the systems used for his comparisons include the Dodge-Romig system, the Phillips SSS, and the systems MIL-STD 105 and AWF. All these are compared and discussed at considerable length and the paper includes numerous tables and graphs with which the author illustrates his statements.

Firstly the author gives an account of the origin and the necessity for the new system and, after stating that it has much in common with the MIL-STD 105A, he proceeds to describe its advantages.

After extensive comparisons with other systems he also compares it with the UDE 0560 Regeln fur Kondensatoren. He states that the presentation varies in each of the systems of sampling but that it is easy to regroup the tests according to individual preference. Tables are also given comparing the VDE 0560 system with the new ASQ-UA6 system.

(Mlle. Gervaise)

MATÉRN, B. (Forest Research Inst., Stockholm) **8.0** (10.0)

Spatial variation. Stochastic models and their application to some problems in forest surveys and other sampling investigations—*In English*

Meddelanden från Statens skogsforskningsinstitut 49-5 (1960) 144 pages (133 references)

The theorey of spatial processes has in recent years come to the fore in a number of important applications, from macrocosmic patterns such as the distribution of galaxies in space, to micro-structures such as the surface pattern of manufactured products, e.g. photographic film, paper, metal; and between these extremes a great many phenomena such as the spatial distribution of plants or animals in the field, of trees in a forest, the pattern of various rock formations on a geological map, the variation of tensile strength in a piece of metal. Matérn gives a review and development by way of a flexible treatment that starts with generalities about processes that are stationary and isotropic (invariant to translation and rotation), continues by a detailed exploration of specific processes at different levels of generality, and makes use of statistical data for a discussion of the realism of the various approaches. The bulk of the paper is devoted to statistical applications, including discussions of the efficiency of random and systematic designs in plane sampling, the estimation of the sampling error from the data of a systematic sample, and specific questions connected with large-scale forest surveys.

Most of the processes considered are constructions based on the Poisson process, as when centres y_i are distributed at random over the plane with constant probability density, and the plane is divided into polyhedric cells Y_i such that all points x of Y_i have y_i for nearest centre. In developing formulae for covariances of integrals, the author studies the distribution of distances between points chosen at random in regions of different kinds. A formula is given for the average surface content (per unit volume) of the boundary of a random set. This average is shown to be proportional to $-c'(0)$ where $c(v)$ is the (supposedly isotropic) covariance-function of the corresponding process. When dealing with data read from maps the author treats the influence of errors of measurements and rounding-off

continued

MATÉRN, B. (Forest Research Inst., Stockholm) **8.0** (10.0)

Spatial variation. Stochastic models and their application to some problems in forest surveys and other sampling investigations—*In English*

continued

Meddelanden från Statens skogsforskningsinstitut 49-5 (1960), 144 pages (133 references)

errors on the observations of a realisation of a process, the effect of " local integration " (e.g. of fertility in field), and the effect of competition between plants, trees, etc.

As to systematic sampling with fixed intensity, it is shown that points in a triangular network give lowest variance among the investigated sampling schemes for the classes of isotropic processes considered. The travel distance connected with different random and systematic sampling schemes is discussed.

(H. O. Wold)

STECK, G. P. & OWEN, D. B. (Sandia Corporation, Albuquerque, New Mexico) **8.8** (8.9)

Percentage points for the distribution of outgoing quality—*In English*

J. Amer. Statist. Ass. (1959) **54**, 689-694 (5 references, 2 figures)

When only a few, perhaps L, lots are going to be purchased, the average outgoing quality limit will be less pertinent for the consumer than the percentage points of the distribution of outgoing quality. The authors derive formulas for these percentage points as functions of incoming quality.

Three cases are considered: the case where the given L lots all have the same known quality; the case where the given L lots have different but known quality; and the case where the L qualities associated with the L given lots constitute a random sample from a distribution with known mean and variance. The first case is handled exactly; the second and third are handled approximately.

(H. T. David)

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BOCK, R. D. (University of Chicago) **9.5** (7.3)

Remarks on the test of significance for the method of paired comparisons—*In English*

Psychometrika (1958) **23**, 323-334 (10 references, 1 table, 2 figures)

A three-component model for comparative judgment which allows for individual differences in preference is proposed. An implication of the model is that errors in the observed proportions due to sampling indivuals in paired comparison experiments are correlated. By neglecting this correlation, Mosteller's test for the method of paired comparisons tends to accept falsely the goodness-of-fit of the Case V solution. It is shown that bounds may be set for the correlation effect which make a valid test possible in some cases and provide useful standard errors for the estimated affective values.

In 1951, Mosteller suggested the use of the arcsine transformation for proportions to test whether the variance of those discrepancies is in excess of that expected from the binomial sampling variability of the observed proportions. When this test is applied to data from moderate-sized samples, it persistently shows that discrepancies are smaller than those expected from sampling variability. That is, the fit of the Case V model appears to be too good rather than too poor.

This paper shows that the anomalous behaviour of Mosteller's test is the result of assuming the sampling errors independent when, in general, they are not. It is concluded that under Case V assumptions, sampling errors for comparisons involving a common object are correlated and on certain assumptions, bounds for this correlation may be set. With a minor alteration, Mosteller's test may be recast in the form of an analysis of variance and the effect of correlation on the variance due to departure from the Case V solution may be derived. It is apparent that the binomial sampling variance used as the error term by Mosteller is in general too large and that the test must frequently fail to detect departure from internal consistency in the Case V solution.

(R. E. Stoltz)

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BEHRENS, W. U. (Kali-Chemie, Hannover) **9.0** (9.7)

The analysis of field experiments according to Mitscherlich—*In German*

Zeit. Acker Pflanzenbau (1959) **109**, 255-290 (11 references, 16 tables, 1 figure)

The paper contains a comprehensive discussion of the adjustment methods due to the so-called Mitscherlich designs, which are systematic designs having the plots in a row with constant order of the treatments in every replication

In the first part the author assumes that the soil fertility is constant for all plots. He computes the estimates of the difference between two treatment effects and of the error variance, without applying any adjustments and applying the moving average and Mitscherlich's adjustment method. As one can expect, all the estimates are unbiased and the variance of the estimated difference may be greater after adjustment than that of the corresponding unadjusted values. Therefore, in this simple case of a constant soil fertility, the adjustment methods are of no value.

In the second part the author investigates the influence of a variation of soil fertility, which he assumes to be a parabola of the ordered plot numbers. Comparing the different adjustment methods he obtains the following results: firstly, without applying any adjustment, the estimates of the difference between two treatments and of the error variance obtained by analysis of variance methods are biased; the bias being dependent on the linear and quadratic term of the soil fertility function and on the two treatments compared. Secondly, after adjustment by the moving average method the estimates are unbiased and can be computed by analysis of variance applied to the adjusted values.

By the adjustment method of Mitscherlich and the regression method of Rundfeldt, the biases are decreased but do not vanish. However, they depend only on the quadratic term of the soil function. The author discusses the problem of which systematic row-design will adjust the soil fertility in the best way if the latter can be expressed by a parabola. He develops his "balanced arrangement", in which the treatments are so arranged that: (1) their average place within the blocks is constant and equals the middle point of the whole design, and (2) the sum of squares of the place numbers within the whole design is constant for all treatments. These conditions can only be fulfilled exactly, if the product $n(p+1)$ is odd (n = number of blocks, p = number of treatments). The author gives some examples of such arrangements.

(B. Schneider)

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DAMARAJU, R. A. (University of Bombay) **9.1** (-.-)

A note on the construction of group divisible designs from hyper-graeco-latin cubes of the first order—*In English*

Bull. Calcutta Statist. Ass. (1959) **9**, 67-70 (7 references)

In this paper the author gives a method of construction of group divisible designs from hyper-graeco-latin cubes of the first order. Consider s^3 treatments arranged in s squares each of side s. Suppose a complete set of (s^2+s-2) mutually orthogonal latin cubes of side s exists. Superimpose all the latin cubes on the arrangement of treatments in the cube formed by s squares. For each latin cube the sets of s^2 treatments corresponding to each of the letters give us s blocks with s^2 treatments in each. Thus $s(s^2+s-2)$ blocks are obtained.

Treatments in a particular row of all the squares give a block of s^2 treatments. The s rows give rise to s blocks.

Another s block can be obtained from the s columns in a similar way, so that all together, from the letters, columns and rows, we generate $s^2(s+1)$ blocks with s^2 treatments in each. Let us consider s^2 treatments contained in a square as a group. There are s such groups. It is easily seen that the $s^2(s+1)$ blocks as derived above provide a group divisible design with the following parameters.

$$b = s^2(s+1),\quad k = s^2,\quad v = s^2,\quad r = s(s+1),\quad \lambda_1 = s,\quad \lambda_2 = s+1$$

The group divisible design corresponding to $s = 3$ is constructed to illustrate the procedure.

(K. R. Shah)

2/365

EISENHART, C. (Statist. Engineering Lab., Nat. Bur. Stand., Washington) **9.0** (9.7)

Some canons of sound experimentation—*In English*

Bull. Int. Statist. Inst. (1960) **37**, III 339-350 (9 references, 3 tables). Discussion abstracted No. **2**/367

Randomisation, replication, and balanced arrangements are presented as fundamental principles of the planning of experiments in general. In the case of comparative experiments, one needs to add to the above mentioned the use of controls and " blind-fold techniques ". These principles are elucidated and illustrated in terms of the 1954 Salk poliomyelitis vaccine field trials. This study consisted of two parts, one planned and carried out in accordance with the above-mentioned principles, and the other without regard to these principles.

The advantages of balanced arrangements in laboratory work are illustrated in terms of a titration experiment devised by Youden. This experiment involves only eight observations, but the structure and scheduling of these is such that, from the results obtained, it is not only possible to determine the concentration of the solution titrated, but also:

(i) to compare the upper and lower halves of the burette to determine the effect of repeated fillings of the burette,

(ii) to compare the two analysts, and

(iii) to determine whether the analytical procedure is subject to a systematic error in the sense of a " blank ".

(C. Eisenhart)

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EISENHART, C. (Report of discussion on a paper by) **9.0** (9.7)
Some canons of sound experimentation—*In English*
Bull. Int. Statist. Inst. (1960) **37**, I 57-59.

Those taking part in the discussion were Hamaker, Box, Martin and Wold.

Hamaker drew attention to the fact that the list of canons of sound experimentation was not complete. He mentioned three examples of canons which equally contributed to the success of an experiment, among those he included the choice and the setting of the levels of the factors.

Box said that while the author stated that the experiments must be designed in advance and not as they proceeded he was sure that he did not mean to imply that the individual experiments, which were often steps in a larger investigation and so became iterative processes, should not occur as such. The object of the statistical methods in these situations was to make the experimental process converge as certainly and as rapidly as possible.

Wold stated that while he acknowledged that the author had made a significant contribution to the classification of the principles of controlled experimentation, some of these principles could be re-stated in formal terms and so further clarified.

The author thanked the participants and said that as the comments were non-controversial and of a constructive nature he did not feel the need for further comment.

(W. R. Buckland)

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HAMAKER, H. C. (Phillips Research Labs., Eindhoven) **9.1** (–.–)
Some examples relating to the design of experiments—*In French*
Rev. Statist. Appl. (1960) **8**, 9-27 (8 tables, 9 figures)

After saying that the design of industrial experiments constitute a field where statistical methods have an important part to play, the author proceeds to present some practical examples to illustrate this statement.

The first example, dealing with power-press swaged joints, states nine factors which have influence on the quality of the assembly and shows how, in practice, all nine factors cannot easily be inspected simultaneously. In these circumstances it is preferable to make a series of simple experiments, so that one can develop a sequential system of experiments. There are various choices as to where these experiments should start and these are next discussed by the author; these include choices from the point of view of production, cost or materials. The author illustrates and describes these experiments.

The second experiment uses fuse links to prove once again how much simpler and quicker it is to use sequential methods. Average results for the fusion time as a function of electric current as well as the maximum and minimum are presented graphically. The third example deals with a steel foundry which produces cylindrical steel magnets in very great numbers. The production difficulties which arise in this particular design and which include the punching of holes and control of the accurate height of the cylinder are discussed and once again a simple series of experiments is drawn up.

Finally, the author states that he has tried to present his examples and conclusions without using an excessive amount of statistical terms: he shows that statistical techniques provide powerful methods for analysing complex situations.

(Mlle. Turlot)

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GRIDGEMAN, N. T. (National Research Council, Ottawa) **9.0** (9.9)

The lady tasting tea, and allied topics—*In English*

J. Amer. Statist. Ass. (1959) **54**, 776-783 (7 references, 1 figure)

The well-known discussion of the principles of experimentation, illustrated by a taste-testing problem, in Fisher's *Design of Experiments*, is the basis of this expository paper. Reappraisal is conducted in the light of the work of Neyman and of Wrighton, which is reviewed briefly. The author supports Fisher's notion of a hypothetical population of experiments. He observes that the null-hypothesis of " no sensory perception " may be tested without constructing a model for an alternative hypothesis; that is, at least from one point of view it is a matter of indifference which of the following is true:

(i) that all ladies are binomially classifiable according to whether or not they can detect the particular tea difference;

(ii) that all ladies may be ranked according to their degree of sensory perception.

It is noted that in many investigations of sensory perception involving only a single subject a marginally detectable stimulus often gives a frequency of correct responses less than 100 per cent. but significantly higher than random expectation. One may therefore postulate a neural mechanism which responds with probability p_s. In the tea-tasting experiment, p_s is taken to be the probability of recognising a cup of either type. Cups which are not recognised may still be assigned correctly by chance. Based on this model the author gives the probability that the subject scores $2R$ successes when confronted with $2N$ cups, known to comprise N of either kind. For $N = 4$ (double tetrad), $\Pr(R)$, $\mathscr{E}(R)$, and Var R are plotted as functions of p_s.

The author distinguishes between " sensitivity " (as used by Fisher in this context) and " efficiency " (in terms of power). Alternative experiments are considered:

(i) duplicate double tetrad;

(ii) quadruplicate double pair;

(iii) eight pair comparisons.

If the lady is allowed one mistake in each, then the sensitivities are equal. Pair comparisons are however most efficient and double tetrad least efficient, uniformly in p_s. Since efficiency depends on the unknown p_s, the

continued

GRIDGEMAN, N. T. (National Research Council, Ottawa) **9.0** (9.9)

The lady tasting tea, and allied topics—*In English* *continued*

J. Amer. Statist. Ass. (1959) **54**, 776-783 (7 references, 1 figure)

author asserts that operationally speaking, sensitivity is specifiable but unimportant, whereas efficiency is important but unspecifiable ".

Fisher's assertion that randomisation is " the physical basis of the validity of the test " is discussed. Misgivings are expressed with regard to the case where inter-cup differences (India tea and China tea) are present, are randomised, but are not balanced between the treatment groups. Finally, the author compares randomisation at the " design " stage with a randomised decision at the " interpretive " stage. The latter introduced to obtain precisely the desired level of significance.

(R. J. Buehler)

KISHEN, K. & SRIVASTAVA, J. N. (Dept. Agric., U.P. and Indian Coun. Agric. Res., New Delhi) **9.2** (9.1)

Mathematical theory of confounding in asymmetrical and symmetrical factorial designs —*In English*

J. Ind. Soc. Agric. Statist. (1959) **11**, 73-110 (10 references, 14 tables)

The methods of finite geometry given by Bose & Kishen [*Sankhya* (1940) **5**, 21-36] for solving the problem of confounding in the symmetrical factorial design s^m, when s is a prime or power of a prime, are extended to the construction of balanced confounded asymmetrical designs, by using curvilinear spaces or hyper-surfaces and suitably truncating the associated Euclidean geometry $EG(m, s)$.

A hyper-surface in $EG(m, s)$ is represented by $a_0+a_1 f_1(x_1)+a_2 f_2(x_2)+\ldots+a_m f_m(x_m) = 0$, $f_i(x) = \Sigma a_{ij}x^j$, a_i and $a_{ij} \in GF(s)$. Taking $f_i(x) = x^{b_t}$, the authors prove three theorems:

(i) If d is a divisor of $s-1 = p^n-1$, then x^d and x^{s-1-d} will give exactly $[(s-1)/d]+1$ distinct values when x varies from a_0 to a_{s-1}.

(ii) The power-matrix $S=(a_r^t)(r, t = a, 1, \ldots s-1)$, where a_r^t are non-zero elements of $GF(s)$, is non-singular and has inverse $S^{-1} = (a_{p-1}^{-1}a_t^{s-r-1})$.

(iii) Let $y = f(x) = a_1x+a_2x^2+\ldots+a_{s-1}x^{s-1}$. There exist a set of matrices such that as x is varies from a_1 to a_{s-1}, only $(k-1)$ distinct values of y are obtained, so that including $x = 0$, we have k distinct levels.

The authors use these theorems for getting a general method of constructing asymmetrical designs. The method is illustrated by constructing the following:

(i) $3\times3\times2$ design in blocks of 6 plots,
(ii) $s^2\times q$ design in blocks of sq plots,
(iii) $s\times q_1\times q_2$ design in blocks of q_1q_2 plots,
(iv) $4\times3\times2\times2$ design in blocks of 12 plots,
(v) $s_1\times s_2\times\ldots\times s_m$ designs in block of $s_1\times s_3\times s_4\times\ldots$ plots,
(vi) $q\times2^2$ and $q\times p^2$ designs, q being any integer, and p odd prime power,
(vii) $(p_1p_2)^m$ designs in blocks of $p_1^{m-v}p_2^{m-v}$ plots.

A connection between balancing in asymmetrical factorial experiments and balanced incomplete block designs is discussed. Some general methods of analysing partially confounded balanced designs are also discussed.

2/371 (C. S. Ramakrishnan)

MITRA, S. K. (Indian Statist. Inst., Calcutta) **9.1** (3.1)

On the *F*-test in the intrablock analysis of a balanced incomplete block design—*In English*

Sankhyā (1960) **22**, 279-284 (4 references)

The paper aims at justifying the use of *F*-test for testing the significance of treatment effects in the intra-block analysis of a balanced incomplete block design.

The author considers the random assignment of blocks of treatments of the design to actual experimental blocks and the further random assignment of treatments to plots within each experimental block. This randomisation induces a distribution for the ratio of sum of squares as given by the statistic (treatment SS)/(treatment SS+error SS). The first two moments of this distribution are obtained under the null hypothesis and compared with the first two moments of the Beta distribution $B(\frac{1}{2}[v-1], \frac{1}{2}[bk-b-v+1])$.

It is shown that for these two distributions, the means are always equal and that the variances are practically equal if $b(k-1)\gg2$; that is, if the number of experimental units is moderately large and the intra-block errors are homogeneous. It is pointed out that the situation is remarkably similar to that of randomised blocks and that the variance ratio distribution provides a reasonably accurate evaluation of the level of significance.

(K. R. Shah)

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MORILLON, Y. (Research Institute of Iron Smelting, Paris) **9.1** (5.1)

The theory of randomisation—*In French*

Rev. Statist. Appl. (1960) **8**, 29-44 (2 references, 5 tables)

This work on the theory of randomisation is based on previous work by Kempthorne [*Design and analysis of experiments* (1952) New York: Wiley, and "Randomisation theory of experimental inference", *J. Amer. Statist. Ass.* (1955) **50**, 946-967] and that done by Kempthorne & Wilk ["Fixed, mixed and random models", *J. Amer. Statist. Ass.* (1955) **50**, 1144-1167].

The author discusses four types of design which have as their common characteristic the utilisation of all the experimental units subject to the operation of randomisation. The study of the distribution of the ratio of mean-squares is simplified under these conditions as certain terms which occur in the decomposition of the sum-of-squares are not random in the null hypothesis.

After stating that the designs he intends to study fall into four groups,

(i) Completely randomised designs,
(ii) Randomised complete blocks,
(iii) Split-plot designs, and
(iv) Latin squares

the author devotes the next section of the paper to a definition of additivity. The following section deals with the direct test for the null hypothesis that there is no difference between treatments and the fourth section is devoted to the basic mathematics necessary to the analysis of variance.

Section five develops mathematically the designs discussed in the first section and is illustrated with several tables in the standard analysis of variance form.

In conclusion the author shows conditions for a valid application of the randomisation test of treatment effects in connection with the four types of designs which are under discussion. It is also shown that randomisation tests are not justified for certain other hypotheses. In the case of the Latin square the various problems merit further study.

(Mlle. Turlot)

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RAO, P. V. (College of Science, Nagpur) **9.1** (-.-)

On the construction of some partially balanced incomplete block designs with more than three associate classes—*In English*

Bull. Calcutta Statist. Ass. (1959) **9**, 87-92 (5 references)

The author defines a partially balanced matrix with parameters α_p, β_p and $\theta_{ipq}(p, q = 1, 2, \ldots, s)$ as a matrix whose elements take integral values 1, 2, ..., s such that:

(i) the integer p occurs α_p times in each row and β_p times in each column,
(ii) there is a partially balanced association scheme, defined on the rows, and
(iii) in any two rows j and j' which are ith associates, integers p and q occur together in θ_{ipq} columns if j and j' are ith associates.

Two balanced incomplete block designs N_1 and N_2 are defined to be associable if:

(i) a one-one correspondence can be set up between the treatments as well as between the blocks of the two designs N_1 and N_2, and
(ii) if new blocks are formed by combining the corresponding blocks of the two designs, any two treatments i and j, i from N_1 and j from N_2, occur together in these new blocks μ times if i and j correspond and η times if they do not.

The author shows that, if in a partially balanced matrix each integer is replaced by the incidence-matrix of one of a set of associable balanced incomplete block designs, the resulting matrix is the incidence matrix of a partially balanced incomplete block design.

(K. R. Shah)

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BASHARIN, G. P. (Moscow) **10.4** (1.5)

The congestion time-limit distribution for a fully available group of trunks—*In Russian*

Teor. Veroyat. Primen. (1960) **5**, 246-252 (9 references, 2 tables)

A fully available group of n trunks is considered under the assumption that a Poisson stream of calls with constant intensity λ is serviced. The full availability group is a loss-system. The holding time is independent of the stream of calls and has an exponential distribution with a mean holding time equal to unity.

Let $\xi(t) = \{\xi_0(t), \xi_1(t), ..., \xi_n(t)\}$ be a random vector, where $\xi_\alpha(t)$ is the life time of the system in its α state ($\alpha = 0, 1, ..., n$) during time interval $[0, t]$. The second moments of the random vector $\xi(t)$ are determined as rational functions of λ. These results make it possible to apply integral and local limit theorems for practical purposes.

(G. P. Basharin)

BLACKWELL, D. (Univ. of California, Berkeley) **10.5** (-.-)

Infinite codes for memoryless channels—*In English*

Ann. Math. Statist. (1959) **30**, 1242-1244 (1 reference)

Consider a memoryless channel for which the input alphabet A and output alphabet B are both finite. For such a channel an infinite code for transmission at rate r is defined to consist of

(i) a sequence of functions f_n $(n>0)$ where f_n maps $[rn]$-tuples of zeros and ones into A,
(ii) a non-decreasing sequence of positive integers M_n satisfying $M_n/n \to 1$ as $n \to \infty$, and
(iii) a sequence of functions g_n $(n>0)$ where g_n maps n-tuples with components in B into M_n-tuples of zeros and ones.

Suppose one is required to transmit an infinite sequence of zeros and ones in code across a channel, say $(x_1, x_2, ..., x_n, ...)$, and that the components of the message are made available in order, one component every r^{-1} units of time. If the transmitter must send one letter every unit of time, the nth letter he sends is $f_n(x_1, x_2, ..., x_{[rn]})$ since only the first $[rn]$ letters of the message are available to him. On the other hand, at the nth unit of time, the receiver has received n letters, $(y_1, y_2, ..., y_n)$ say, and estimates the first M_n letters $(x_1, x_2, ..., x_{M_n})$ of the original message to be $g_n(y_1, y_2, ..., y_n)$. For the rate $r = 1$, the above discussion is greatly simplified, and the author states and proves his theorem only for this case.

The theorem of this paper is that if the capacity of the channel exceeds 1, then there exists an effective code. A code is said to be *effective* if for every message $(x_1, x_2, ..., x_n, ...)$ consisting of zeros and ones, only a finite number of the receiver's estimates $g_1(y_1), ..., g_n(y_1, ..., y_n), ...$ are incorrect. The author proves the theorem by the use of Shannon's exponential error bounds and states that while it would be desirable to extend the theorem to finite-state channels, the bounds are not yet known for general finite-state channels.

(R. Pyke)

BLACKWELL, D., BREIMAN, L., & THOMASIAN, A. J. (University of California, Berkeley) **10.5** (10.1)
The capacity of a class of channels—*In English*
Ann. Math. Statist. (1959) **30**, 1229-1241 (14 references)

This paper generalises the definition of capacity of a discrete memoryless channel to the case of a class of such channels Γ. This capacity $C = \sup_Q \inf_\gamma R_\gamma(Q)$, where $R_\gamma(Q)$ is the usual transmission note defined as the information content of the source with input probability Q minus the equivocation of one of the channels γ in Γ. Now $R \geqq 0$ is an attainable note for the class Γ if there exists a sequence of codes $(e^{nR}, \varepsilon_n, n)$ for the class with $\varepsilon_n \to 0$. We say $(e^{nR}, \varepsilon_n, n)$ is a code for each n-extension of each channel in Γ if it is a one to one map from $[e^{nR}]$ sets of words of length n in the receiving alphabet into words of length n encoded in the transmitting alphabet such that the maximum probability of error using the code is of size ε_n.

The main result is that the supremum of the set of attainable notes for a class of discrete memoryless channels is equal to the capacity of that class and the method of proof gives an exponential error bound for any attainable note $R < C$. This result is demonstrated through a series of steps interesting *per se*. A basic inequality [see Feinstein, *Foundations of Information Theory* (1958) New York: McGraw Hill] gives the same bound for a single discrete channel for the maximum probability of error that Shannon [*Information and Control* (1957) **1**, 6-25] had given for the average probability of error. This inequality permits a simple proof of the direct half of Shannon's Theorem, that the capacity of a discrete memoryless channel is not greater than the supremum of attainable rates.

The third theorem then gives an exponential bound on the error of a code for a single channel which depends only on the number of elements in the transmitting and receiving alphabets and $C-R$. This is then generalised to the case of a finite number L of channels in a class in lemma 3 and theorem 4. The underlying idea being to consider the mixture, with probability $1/L$, of the L channels in the finite class as a channel and to replace the error bound ε by ε/L.

It is then shown that for fixed transmitting and receiving alphabets there is a large finite number of

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BLACKWELL, D., BREIMAN, L., & THOMASIAN, A. J. (University of California, Berkeley) **10.5** (10.1)
The capacity of a class of channels—*In English*
Ann. Math. Statist. (1959) **30**, 1229-1241 (14 references)
continued

channels such that any channel on these alphabets is close, in several senses, to one of this number. Lemma 5 shows that if a channel has a sequence of codes $(e^{nR}, \varepsilon_n, n)$ with $\varepsilon_n = e^{-nB}$ for large n with $B>0$ then this same sequence of codes can be used for all channels in a certain neighbourhood, determined by one of the measures of closeness in lemma 4, of the channel. This explains the attention to exponential error bounds.

The direct half of the first theorem is proved by approximating the class Γ of channels by a certain finite set of channels from lemma 4, obtaining an exponential error bound for this finite set by theorem 4 and using lemma 5 to show that such a code must work also for Γ. The converse half is then proved with lemma 6 by a variation of a result due to Fano.

(S. C. Saunders)

CHERKASOFF, I. D. (Murmansk) **10.1** (-.-)

Criteria for co-ordinate homogeneity continuous Markoff processes—*In Russian*

Teor. Veroyat. Primen. (1960) **5**, 229-237 (10 references)

The author considers a Markoff process with an infinitesimal operator

$$A_t = A^i(t, x)\frac{\partial}{\partial x^i} + B^{ij}(t, x)\frac{\partial^2}{\partial x^i \partial x^j}$$

where $x = (x^1, \ldots, x^n)$ is a point in Riemann space V_n with a metric $g_{ij}(t, x)$. The necessary and sufficient conditions for the existence of transformation

$$x'^i = x'^i(t, x^1, \ldots, x^n)$$

which transforms the operator A_t into the well-known operator

$$A_t^0 = B^{i'j'}(t)\frac{\partial^2}{\partial x^{i'} \partial x^{j'}}$$

is given.

A statistical example is given in which these conditions are applied for establishing the density of the probability $f(t, x;\ \tau, \xi)$ of a certain Markoff process.

(I. D. Cherkasoff)

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DAS, S. C. (Ravenshaw College, Cuttack, India) **10.1** (-.-)

A note on the structure of a certain stochastic model—*In English*

Sankhyā (1960) **22**, 345-350 (3 references)

The author considers the following stochastic model. There are $(m-1)$ types of successes with the following properties: the occurrence of the rth type of success in any trial excludes for the next r trials the occurrence of any type of success. After this period the probability of having a success of the jth type is $p\alpha_j$ ($j = 1, 2, \ldots, m-1$) where $\sum_{j=1}^{m-1} \alpha_j = 1$ and p is the probability of having any one of the $m-1$ types of successes.

He identifies this as a Markoff chain consisting of the states $E_0, E_1, \ldots, E_{m-1}$ where E_0 is the state where a success can occur and E_j is the state where a success cannot occur for the next $m-j$ trials. The system moves from E_0 to $E_0, E_1, E_2, \ldots$ or E_{m-1} and from E_j to E_{j+1} for $j = 1, 2, \ldots, m-1$ ($E_m = E_0$).

The author finds the stationary or the limiting probabilities for this Markoff chain and evaluates the mean and variance of the recurrence times for different states.

As an application the author mentions the study of accident data in situations where a worker getting involved in an accident and hospitalised for a certain time depending on the type of accident, is immune to further accidents during that period.

(S. R. S. Varadhan)

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DANSKIN, J. M. (Rutgers University, New Jersey, U.S.A.) **10.5** (0.0)

The theory of reconnaissance—*In English*

WADC Technical Note 59-409 (1959) Aero. Res. Labs., Wright-Patterson Air Force Base iv+31 pp. (1 reference)

A study is made of the optimum distribution of aerial reconnaissance effort against land targets in the presence of decoys. The model considered is one in which the reconnoitering forces allocate effort among various regions, their objective being the location of the targets, assuming that the side being reconnoitered is passive. The model is described in terms of photographic reconnaissance, but the theory is equally applicable to other methods of reconnaissance; visual, radar, or infra-red, for example, since no specific properties of photographs are used. The basic data assumed as given for the model are the probabilities p_{ij} that an object of kind i gives a photograph of type j. These appear, vary and vanish as the level of reconnaissance changes. We shall be concerned with situations in which there is no specific military attacking system whose action is associated with the results of the reconnaissance.

The results of the reconnaissance are indicated by pins stuck in a map, which is then regarded with great interest but without specific action by the appropriate commander. Associated with each map will be an uncertainty. We shall assume that the reconnaissance command is searching for information concerning missile installations and decoys, both of which are supposed to be of a certain specified type. We shall suppose that the command has divided the area of interest into regions, in each of which there are just three possibilities: either there is nothing in the region; or there is a decoy in the region; or there is a missile installation in the region. Each pin indicates the apparent situation in the region by some device equivalent to the giving of three numbers, p_1, p_2, and p_3, the probabilities that the object is nothing, a decoy, or a missile installation. We have of course $p_1+p_2+p_3 = 1$. We suppose that a target of kind i produces a photograph of type j during a reconnaissance with probability p_{ij}. The probability that a photograph of type j is produced is then $q_j = \sum_i p_i p_{ij}$.

The information function of the theory of communication is chosen as the measure of effectiveness.

continued

DANSKIN, J. M. (Rutgers University, New Jersey, U.S.A.) **10.5** (0.0)

The theory of reconnaissance—*In English*

continued

WADC Technical Note 59-409 (1959) Aero. Res. Labs., Wright-Patterson Air Force Base iv+31 pp. (1 reference)

The information of a reconnaissance is defined as the change in the uncertainty of the map resulting from that reconnaissance. The expected information produced by the reconnaissance is shown to be

$$I(x) = c\sum_i\sum_j p_i p_{ij} \log(q_{ij}/p_i),$$

where x denotes the level of reconnaissance and c is a constant. It is proved that under certain reasonable assumptions the information function is increasing and first convex, then later concave. The problem is thus reduced to a familiar one in operations research, the maximisation of the sum of several convex-concave functions subject to linear side conditions.

(H. L. Harter)

DOBRUSHIN, R. L. **10.1** (–.–)

Properties of sample functions of stationary Gaussian processes—*In Russian*

Teor. Veroyat. Primen. (1960) **5**, 132-134 (4 references)

Let $\xi_t(\omega)$, $(|\,t\,|<\infty)$ be a separable stationary Gaussian process with a continuous correlation function. Then the following alternative holds: either (i) for almost all ω, sample functions of process $\xi_t(\omega)$ are continuous functions of t or (ii) there exists a $\beta>0$ such that, for almost all ω, sample functions $\xi_t(\omega)$ are such as

$$\overline{\lim_{t\to t_0}}\, \xi_t(\omega) - \underline{\lim_{t\to t_0}}\, \xi_t(\omega) \geqq \beta$$

for any t_0. In the second case almost all sample functions do not have points of first order discontinuity.

(R. L. Dobrushin)

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FINCH, P. D. (London School of Economics) **10.4** (1.5)

A probability limit theorem with application to a generalisation of queueing theory—*In English*

Acta Math. Acad. Sci. Hung. (1959) **10**, 317-325 (3 references)

The following single-server queueing system is considered: the nth customer arrives at the instant τ_n and his service time is s_n, $n = 1, 2, \ldots$. If he finds the server idle, then, before starting service he waits a time v_n, otherwise he commences service as soon as the $(n-1)$th customer finishes it.

Let w_n be the waiting time of the nth customer, then

$$w_{n+1} = \begin{cases} w_n+u_n & \text{if } w_n+u_n>0 \\ v_{n+1} & \text{if } w_n+u_n \leqq 0 \end{cases}$$

where $u_n = s_n+\tau_n-\tau_{n+1}$, $n = 1, 2, \ldots$. Let $W_n(x) = \Pr(w_n \leqq x)$ and $W(x) = \lim_{n\to\infty} W_n(x)$. It is proved that under certain conditions $W(x)$ exists and satisfies an integral equation. In the case where the input process is of Poisson-type further discussions are given.

(G. Bánkövi)

FINCH, P. D. (London School of Economics) **10.4** (1.5)

Balking in the queueing system GI/M/1—*In English*

Acta Math. Acad. Sci. Hung. (1959) **10**, 241-247 (4 references)

A single-server queueing system is considered with balking; that is to say a customer joins the queue consisting of m customers with probability b_m; the sequence $\{b_m\}$ is non-increasing, $b_0 = 1$, $b_k = 0$ $(k \geq N+1)$. A random variable $\eta(t)$ is defined as follows: $\eta(t) = k$, $k = 0, 1, \ldots, N+1$ if at time t there are k customers present. Let $\eta_n = \eta(\tau_n - 0)$, where τ_n denotes the instant of arrival of the nth customer. Let $P_k = \lim_{n\to\infty} \Pr(\eta_n = k)$ and $P_k^* = \lim_{t\to\infty} \Pr(\eta(t) = k)$ $(k = 0, 1, \ldots, N+1)$.

The author proves that under certain conditions concerning the input and the service processes, the limiting distributions P_k and P_k^* exist and are independent of the initial state; expressions for P_k and P_k^* are given.

(G. Bánkövi)

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FINCH, P. D. (London School of Economics) **10.4** (1.5)

On the distribution of queue size in queueing problems—*In English*

Acta Math. Acad. Sci. Hung. (1959) **10**, 327-336 (8 references)

An m-server queueing system is discussed by the author in this paper. A random variable $\eta(t)$ is defined as follows: $\eta(t) = k$, $k = 0, 1, 2, \ldots$, if at time t there are k customers present. Let t_n be the instant of arrival, s_n the service time and w_n the waiting time of the nth customer. Let $\eta_n = \eta(t_n - 0)$ and $\xi_n = \eta(t_n + w_n + s_n + 0)$. The probabilities $P_k(t)$, $Q_k(n)$, $R_k(n)$, $k = 0, 1, 2, \ldots$, are defined by the equations

$$P_k(t) = \Pr(\eta(t) = k), \quad Q_k(n) = \Pr(\eta_n = k), \quad R_k(n) = \Pr(\xi_n = k).$$

The author proves five limiting theorems concerning $\eta(t)$, $P_k(t)$, $Q_k(n)$ and $R_k(n)$.

(G. Bánkövi)

FIRESCU, D. (University of Bucharest) **10.2** (4.8)

Estimation functions for the fundamental probabilities of a multiple stationary Markoff chain with a finite set of states—*In French*

Bull. Math. Soc. Sci. Math. Phys. R.P.R. (1958) **50**, 401-410 (5 references)

Let us consider an s-multiple stationary Markoff chain with a finite set of states and suppose that the stochastic matrix of the fundamental probabilities of the given chain belongs to the regular positive case in the sense of Fréchet.

First some theorems of multi-dimensional asymptotic normality are proved for the random variables corresponding to the number of occurrences of a succession of states in a sample of given size. Next it is shown that the considered estimation functions for the unknown fundamental probabilities of the chain are correct as well as Gaussian for the case when either all these probabilities are unknown or refer exclusively to one or more complete rows of their matrix: the respective estimation functions are asymptotically efficient. Analogous properties are also proved for the estimation functions for the inverse fundamental probabilities of the stationary chain of the family of the given chain.

(R. Theodorescu)

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GANI, J. (University of W. Australia, Nedlands) **10.1** (–.–)

Time dependent results for a dam with ordered Poisson Inputs—*In English*

Nature (1960) **188**, 341-342 (6 references)

The content of a dam at time t, when fed by an input $X(t)$ and subject to a steady release, may be defined

$$Z(t) = Z(t+\delta t) - \delta X(t) + (1-\eta)\delta t$$

where $\eta\delta t$ $(0 \leqq \eta \leqq 1)$ is the time during t, $t+\delta t$ that the dam is empty. The input is a Poisson process (with parameter λ) with respect to time but each input is of a constant size, α_1 and α_2 and always arriving in that order. The process $Z(t)$ is Markoffian when the number of inputs up to t is known. Using the device of the kronecker delta, the author writes down the transition distribution function of the dam content and shows that it satisfies differential-difference equations similar to those used by Tackács [*Acta Math. Acad. Sci. Hung.* (1955) **6**, 101].

The Laplace-Stieljes transforms of the differential-difference equations lead to integral equations involving the probabilities of emptiness and which are not readily solved. Explicit results may be obtained however, by methods of direct probability. The author draws attention to some of his related work (to be published) where he obtains probabilities of first emptiness recursively from a modified form of an equation used by D. G. Kendall [*J. R. Statist. Soc.* B (1957) **19**, 207-233]. A simpler method of evaluation using truncated generating functions is also given [see also *Math. Ann.* (1958) **136**, 454].

Combinatorial arguments are used to establish the probabilities of first and second emptiness. Where $\alpha_1 = \alpha_2$ the transition distribution function reduces to a form given by Gani & Prabhu [*J. Math. Mech.* (1959) **8**, 658]. The method outlined in this work is equally applicable to the case of $p > 2$ ordered inputs, $\alpha_1, \ldots, \alpha_p$.

(W. R. Buckland)

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GRIFFING, B. (Div. of Plant Industry, CSIRO, Canberra) **10.9** (2.8)

Theoretical consequences of truncation selection based on the individual phenotype—*In English*

Aust. J. Biol. Sci. (1960) **13**, 307-343 (10 references)

Theoretical consequences of truncation selection based on the individual phenotype are examined for the following cases of increasing genetic complexity:

(i) an arbitrary number of alleles at a single locus,
(ii) an arbitrary number of alleles at each of two linked loci, and
(iii) a completely general genetic situation.

The analyses are facilitated by generalising the concept of hereditary units to include not only the gene but also units of higher levels of organisation.

Analyses based on the gamete as the basic unit of inheritance, but with a gene interpretation, permit a detailed examination of the consequences of selection and relaxation following selection for the two-locus case. It is shown that the immediate response to selection may be different from that predicted on the basis of gene analysis if an additive × additive type of epistasis occurs. However, due to the "mutability" of these higher order inheritance units, the population mean, on relaxation of selection, decays to that predicted by the gene analysis approach.

Finally, by using the individual as the unit of inheritance it is possible to make certain statements for very general genetic situations. These statements are formulated in terms of covariances among relatives.

(B. Griffing)

HANANIA, Mary I. (University of California, Berkeley) **10.1** (10.3)

A generalisation of the Bush-Mosteller model with some significance tests—*In English*

Psychometrika (1959) **24**, 53-68 (5 references, 2 tables)

The authoress proposes a modification of the Bush-Mosteller learning model to test the hypothesis that reward and non-reward are equally effective in promoting learning in a continuous reinforcement situation. The hypothesis is reduced to a single parameter, although other parameters remain unspecified.

The Bush-Mosteller model is explored and the modification is statistically developed with the proposed test having the property of being asymptotically locally most powerful among all tests of the same size and asymptotically similar. The test is illustrated in the testing of the single and alternate hypotheses.

(G. L. Hines)

HANS, O. (Czechoslovak Academy of Sciences, Prague) **10.0** (0.4)

Measurability of extensions of continuous random transforms—*In English*

Ann. Math. Statist. (1959) **30**, 1152-1157 (8 references)

In this paper the author gives probabilistic versions of the Tietze and Hahn-Banach extension theorems. Let $(\Omega, \mathfrak{S})$ be a measurable space. In the case of the first theorem mentioned above, it is shown that if V is a random transform of $\Omega \times M$ into the reals for M, a closed subset of a separable metric space X; and if V is continuous for each ω with $|V(\omega, x)| \leq s(\omega)$, then there is a random transform T of $\Omega \times X$ into the reals which agrees with V on $\Omega \times M$ and is continuous for each ω with $|T(\omega, x)| \leq s(\omega)$.

In the case of the latter theorem, it is shown that if V is a random transform of $\Omega \times M$ into the reals for M a linear manifold of a separable real normed linear space X, and if V is a bounded linear functional for each ω with norm $s(\omega)$, then there is a random transform T of $\Omega \times X$ into the reals which agrees with V on $\Omega \times M$, and is a bounded linear functional for each ω with norm $s(\omega)$. The second theorem remains true when X is a non-separable Hilbert space and M is a subspace of X. The proofs of the theorems depend on a construction argument and thereafter a proof of measurability.

(N. D. Ylvisaker)

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HASMINSKY, R. Z. (Moscow) **10.1 (0.1)**

Ergodic properties of recurrent diffusion processes and stabilisation of the solution to the Cauchy problem for parabolic equations—*In Russian*

Teor. Veroyat. Primen (1960) **5**, 196-214 (22 references)

In this paper the existence of a unique invariant measure for Markoff processes satisfying some (in number 9) conditions is proved. This result is applied to obtain the asymptotic properties of the solution to the Cauchy problem for the parabolic equation $\frac{\partial u}{\partial t} = Lu$ when $t \to +\infty$. It is established that these properties depend on properties of the solution of the external Dirichlet problem for the equations $Lu = 0$ and $Lu = -1$. The sufficient conditions for them, expressed in terms of the behaviour of the coefficients in the equation $Lu = \frac{\partial u}{\partial t}$, are given in the appendix.

(R. Z. Hasminsky)

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IADRENKO, M. I. (Ukrainiane) **10.0** (-.-)

Isotropic Gauss random fields of the Markoff type on a sphere—*In Russian*

Dopov. Acad. Nauk Ukrain. SSR (1959) 231-236 (7 references)

The author describes the random fields, indicated in the heading, on a sphere S_m in $(m+1)$-dimensional space and on a sphere S_∞ in Gilbert space possessing, in addition, some property of "Markoffness". In the case of $m = 2$ this property reduces to the requirement that whatever the curve K dividing S_2 into two parts may be and whatsoever the points P_1 and P_2 separated by K, the random function $\xi(P_1)$ and $\xi(P_2)$ are independent if the values $\xi(P)$ on K are known.

(M. I. Iadrenko)

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KATZ, L. & PROCTOR, C. H. (Michigan State University) **10.1** (0.6)

The concept of configuration of interpersonal relations in a group as a time-dependent stochastic process—*In English*

Psychometrika (1959) **24**, 317-327 (3 references, 7 tables)

To explain changes in the sociometric configuration of a group to time, the problem arises of the extent to which such changes may be viewed as the aggregation of part-processes occurring at the level of two-person choice structures. A possible model is a Markoff chain in which three possible states are mutual choice, one-way choice, and indifference; one realisation for each pair of choosing individuals in the group. Choice data for an eighth-grade classroom are fitted to this model and are used to answer questions of constancy of transition probabilities, order of the chain and sex differences.

Within the limits of the number of cases available no evidence was found to suggest that the transition probabilities varied with time. For the two-month gaps there may be second-order dependence of the chain, but the chain of four-month gaps did not depart significantly from first-order dependence. No departures from the first-order Markoff chain model achieves statistical significance among the within-sex choices.

(R. E. Stoltz)

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KATZ, M., Jr. & THOMASIAN, A. J. (Univ. of Chicago and Univ. of California, Berkeley) **10.1** (1.6)

An exponential bound for functions of a Markoff chain—*In English*

Ann. Math. Statist. (1960) **31**, 470-474 (9 references)

Let X_1, X_2, X_3, ... be a finite r-state ($r \geqq 2$) ergodic Markoff chain with arbitrary initial distribution and $r \times r$ stationary transition matrix $\boldsymbol{P}$ with p denoting the smallest non-zero element of $\boldsymbol{P}$. Let $p_1, p_2, \ldots, p_r$ be the stationary distribution for $\boldsymbol{P}$. Assume further that there is only one ergodic class of states $E \subset R = \{1, 2, \ldots, r\}$.

Consider any real-valued function f defined on R. Let $S_n = \sum_{k=1}^{n} f(X_k)$, $\mu = \sum_{k=1}^{r} p_k f(k)$, and $M = \max_{k \in R} f(k) - \min_{j \in R} f(j)$. The authors use recurrent event theory in a manner analogous to Chung [*Trans. Amer. Math. Soc.* (1954) **76**, 397-419] and Doblin [*Bull. Soc. Math. France* (1938) **52**, 210-220] to prove the following theorem concerning the convergence of $n^{-1} S_n$ to μ:

Let m be a positive integer and let $\varepsilon > 0$. Then

$$\Pr\{|n^{-1}S_n - \mu| \geqq \varepsilon \text{ for some } n \geqq m\} \leqq 2Ae^{-B\varepsilon^2 m}$$

$$\Pr\{S_n \geqq n(\mu+\varepsilon) \text{ for some } n \geqq m\} \leqq Ae^{-B\varepsilon^2 m}$$

$$\Pr\{S_n \leqq n(\mu-\varepsilon) \text{ for some } n \geqq m\} \leqq Ae^{-B\varepsilon^2 m}$$

where

$$A = 8r/p^r(1-e^{-B\varepsilon^2}), \quad B = p^{3r}/2^8 M^2 r^2.$$

Since the transition matrix $\boldsymbol{P}$ is permitted to have zero entries the theorem can be applied equally to multiple Markoff chains; that is, to sums of the form $S_n = \sum_{k=1}^{n} f(X_k, X_{k+1}, X_{k+2})$.

The article closes with an example involving a countable state space Markoff chain, which otherwise satisfies the hypothesis of the theorem, and a real-valued function defined on the positive integers for which there exist no constants A and B such that

$$\Pr\{S_n \geqq n(\mu+\varepsilon)\} \leqq Ae^{-B^2 m}.$$

Bounds for the one-sided inequalities are also given. Some possible applications for such bounds can be seen in Chernoff "A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations" [*Ann. Math. Statist.* (1952) **23**, 493-507].

(W. L. Nicholson)

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KOLMOGOROFF, A. N. & ROZANOFF, Y. A. (Moscow) **10.1** (10.6)

On a strong mixing condition for stationary random Gaussian processes—*In Russian*

Teor. Veroyat. Primen. (1960) **5**, 222-227 (9 references)

In this paper the authors consider conditions which guaranteed the strong mixing of stationary random process $\xi(t)$. It is proved, for example, that if the spectral density $f(\lambda)$ of the process $\xi(t)$ is continuous and positive (parameter t is discrete); or $f(\lambda)$ is positive and uniformly continuous and for large λ

$$\frac{m}{\lambda^k} \leqq f(\lambda) \leqq \frac{M}{\lambda^{k-1}}$$

where parameter t is continuous, then strong mixing is the case.

(Y. A. Rozanoff)

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KAWATA, T. (Math. Dept., Tokyo Institute of Technology) 10.6 (1.6)

Some convergence theorems for stationary stochastic processes—*In English*

Ann. Math. Statist. (1959) **30**, 1192-1214 (13 references)

Let $\mathscr{E}(t)$, $-\infty<t<\infty$ be a continuous stationary process of the second order, in the wide sense, with means zero and spectral representations

$$\mathscr{E}(t)=\int_{-\infty}^{\infty} e^{it\lambda}dZ(\lambda),\quad \rho(u)=\int_{-\infty}^{\infty} e^{iu\lambda}dF(\lambda).$$

The first integral is a stochastic integral. The second is an ordinary integral, with $\rho(u)$ the covariance function of the $\mathscr{E}(t)$ process and $F(\lambda)$ the spectral distribution of the $\mathscr{E}(t)$ process. If $L(s)$, a real-valued function of bounded variation on each finite interval, has an L_2 (with respect to dF) Fourier transform $k(x)$, that is

$$\lim_{\substack{A\to-\infty\\ B\to\infty}}\int_{-\infty}^{\infty}\left|\int_{A}^{B} e^{ixs}dL(s)-k(x)\right|^2 dF(x)=0$$

then a process $\eta(t)$ is defined by

$$\eta(t)=\int_{-\infty}^{\infty}\mathscr{E}(t-s)dL(s).$$

Here the integral is defined first over finite intervals and then a mean-square limit taken. The process $\eta(t)$ is stationary with spectral distribution $|k(x)|^2dF(x)$. If in addition a real valued function $f(t)$ is given such that $\int_{-\infty}^{\infty}|f(t-s)|\,dL(s)<\infty$ for all real t then set $X(t)=f(t)+\mathscr{E}(t)$ and define

$$\int_{-\infty}^{\infty}X(t-s)dL(s)=\eta(t)+\int_{-\infty}^{\infty}f(t-s)dL(s).$$

For a real valued function $K(s)$, the author considers convergence theorems for $n\int_{-\infty}^{\infty}X(t-s)K(ns)ds$ (as $n\to\infty$) similar to the classical theorems for the non-random case $\mathscr{E}(t)\equiv 0$. For example, Theorem 3.2 in the paper states that if $f(s)[1+|s|^{\frac{3}{2}}]^{-1}$ and $[1+|s|]K(s)$ are in $L_1(-\infty,\infty)$, if $f(t+u)-f(t)=O(u)$ as $u\to 0$, and if $K(s)$ is bounded and $K(s)=o[|s|^{-\frac{3}{2}}]$ as $s\to\infty$ then

$$\mathscr{E}\left|n\int_{-\infty}^{\infty}X(t-s)K(ns)ds-X(t)\int_{-\infty}^{\infty}K(s)ds\right|^2=o\left(\frac{1}{n}\right).$$

Superimposing a random element on a Wiener-type formula of Bochner, the author obtains his theorem 4.2 which states that if $K(x)$ is absolutely continuous in

continued

KAWATA T. (Math. Dept., Tokyo Institute of Technology) 10.6 (1.6) *continued*

Some convergence theorems for stationary stochastic processes—*In English*

Ann. Math. Statist. (1959) **30**, 1192-1214 (13 references)

every finite interval with $x^2|K(x)|$ uniformly bounded, if $T^{-1}\int_{-T}^{T}|f(t)|\,dt$ is uniformly bounded and if

$$\lim_{T\to\infty}T^{-1}\int_{-T}^{T}f(t)e^{-i\xi t}dt$$

exists for some ξ then

$$\lim_{a\to 0}\int_{-\infty}^{\infty}X(t)e^{-i\xi t}aK(at)dt$$
$$=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}X(t)e^{-i\xi t}dt\int_{-\infty}^{\infty}K(t)dt$$

the limits existing and being taken in the mean of order two.

The author then considers the behaviour for large T of

$$J(T)=\frac{1}{4\pi T}\left|\int_{-T}^{T}\mathscr{E}(t)e^{-ixt}dt\right|^2$$

which, if $\mathscr{E}(t)$ were not a process but an ordinary trigonometric polynomial, would be the ordinary periodogram. Under additional hypotheses on the $\mathscr{E}(t)$ process which insure that it behaves like a Gaussian process, he proves that

$$\lim_{T'>T\to\infty}\mathscr{E}\,|J(T)-J(T')|^2-\left(1-\frac{T}{T'}\right)2p^2(x)=0$$

if $x\neq 0$. The term $2p^2(x)$ is replaced by $4p^2(0)$ if $x=0$. Here $p(x)$ is the density function (assumed to exist and to be bounded and continuous) of $F(x)$. There is a similar theorem on the behaviour of covariance $[J(T),J(T')]$ for large T and T'.

(R. Blumenthal)

LESLIE, P. H. (Bur. Animal Pop., Depart. Zoological Field Studies, Oxford) **10.1** (10.9)

A note on some approximations to the variance in discrete-time stochastic models for biological systems—*In English*

Biometrika (1960) **47**, 196-197 (2 references)

This paper is relevant to a description by Leslie [*Biometrika* (1958) **45**, 16-31] of a discrete-time stochastic model for studying the properties of certain biological systems by numerical methods wherein certain empirical approximations were proposed to the variance of the " number in existance " at time $t+1$ conditional on the state of the system at time t.

This current paper proposes a more general approximation which is at least as good as the original and which has the advantage of being applicable to wider ranges for the values of the birth and death rates to be found in the mathematical description of the stochastic model.

Such approximations save considerable machine time in the performance of, say, a numerical experiment with two interacting species since the necessity of including a logarithmic sub-routine in the programme is avoided.

(J. G. Saw)

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LEUND, C. S. (Moscow) **10.1** (-.-)

On conditional Markoff processes—*In Russian*

Teor. Veroyat. Primen. (1960) **5**, 227-228 (4 references)

The author considers a pair of random processes X_t, Y_t, which jointly form a Markoff process Z_t. The conditional distribution of the process y_t, for the condition of known realisation of process X_t during some time interval, is examined. Dynkin has proposed that if X_t is a Markoff process, the conditional distribution of Y_t will satisfy a functional equation similar to the known Kolmogoroff-Chapman equation.

The author has proved this proposition, but details of the proof are omitted in this paper.

(C. S. Leund)

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LUM, Mary D. (Wright Air Develop. Center, Dayton, Ohio) **10.5** (1.0)

A comparison of an error-correcting eight-unit with other teleprinter codes for binary transmission and ternary reception—*In English*

WADC Technical Note 58-231 (1959), Aero. Res. Labs., Wright-Patterson Air Force Base vii+36 pp. (2 references, 4 tables, 6 figures)

Teleprinter information is transmitted electrically, using a sequence of " marks " and " spaces ". In true binary transmission, a mark represents a pulse sent on one frequency f_1, while a space represents a pulse sent on another frequency f_2. With this type of transmission, the decoder can make three possible decisions (ternary reception): firstly, accept pulse on f_1, secondly, accept pulse on f_2, or finally, accept neither the pulse on f_1 nor that on f_2. This report is concerned with ternary reception of binary transmission. Binary reception can be considered a special case of this more general situation.

As used in this report " error-detection " means the ability to detect at least one bit-error in a received character, " error-correction " means the ability to detect and correct at least one bit-error in a received character.

A study is made of an eight-unit " four marks plus four spaces " binary code in comparison with the following six binary codes: a seven-unit " four marks plus three spaces " code with shifts; a six-unit code; a five-unit code with shifts; a seven-unit parity check code; a six-unit parity check code with shifts; and an eight-unit double parity check code. The common feature of these codes is that they can be made compatible with the existing 5-unit Baudot code since they all contain at least 62 characters. The purpose is to determine whether the proposed eight-unit code is optimum (with respect to minimal susceptibility to character errors) for various bit-error rates.

Formulas are derived for the probability of character errors for the different codes as functions of the proportion of bit-errors. Tables and graphs of these probabilities versus proportion of bit-errors for the range of 10 per cent. missing bits to 1 per cent. missing bits are given for two cases: firstly, missing bits only; and secondly, ten per cent. as many wrong bits as missing bits. An Appendix gives a list of the type of characters which are error-corrected or error-detected by each individual code.

The results indicate that the eight-unit double parity check code is definitely the best of the seven codes studied.

2/401 (Mary D. Lum)

McGREGOR, J. R. (University of Birmingham) **10.7** (2.7)

An approximate test for serial correlation in polynomial regression—*In English*

Biometrika (1960) **47**, 111-119 (10 references, 1 table)

When the error terms in the regression model are serially correlated the usual least-square procedure fails. Durbin & Watson [*Biometrika* (1950) **37**, 409-428 and (1951) **38**, 159-178] developed bounds to the significance points of a test criterion d, defined as the ratio of the sum of squares of successive differences of residuals to the sum of squares of residuals from a fitted regression.

In this paper, McGregor obtains a Pearson Type I approximation to the distribution of d and observes that the relative error between the true and approximate density functions is of order $1/n^2$ where n is the sample size. In addition for the case of regression on orthogonal polynomials, it is shown in an appendix that the exact distribution of d differs from that of d_u (the upper bound given by Durbin & Watson) by order $1/n^2$.

A table is given comparing 1 per cent. and 5 per cent. points of d_u with those of the approximate (Type I) distribution of d for various values of the sample size and orders of the polynomial regression.

The proof of the results rests on the properties of a certain determinant and on an argument due to Daniels [*Biometrika* (1956) **43**, 169-185].

(J. G. Saw)

2/402

MALECOT, G. **10.1** (–.–)

The problem of Brownian movement and its generalisations—*In French*

Publ. Inst. Statist. Paris (1959) 7, 186-195

In this paper the author describes two generalisations of the " random walk " process.

The first consists of the supposition that the possible jumps, instead of being of constant amplitude, possess algebraic values X, with respective probabilities π_1.

The second consists of the admission that the times of these jumps are random variables.

(Mlle. Gervaise)

MARYANOVICH, T. P. **10.4** (2.5)

Generalisation of Erlang's formulae in the case when the lines may be broken and renewed—*In Russian*

Ukrain. Mat. Z. (1960) **12**, 279-286 (3 references)

A fully available group of n trunks is considered under the assumption that a Poisson stream of calls with constant intensity is serviced. The author considers a loss-system: the lines may be broken in the moments t_i and renewed in the time η; $\Pr\{\eta<x\} = G(x)$, $\Pr\{t_{i+1}-t_i<x\} = H(x)$, $\Pr\{\zeta<x\} = F(x)$, ζ is the service time.

Two following conditions are satisfied:

$$\int_0^\infty [1-G(x)]dx<\infty, \int_0^\infty [1-F(x)][1-H(x)]dx<\infty.$$

The stationary distribution p_{ij}, where i is the number of busy lines, j the number of broken lines, and $i+j\leqq n$ is given.

(B. Gnedenko)

MASUYAMA, M. (Tokyo University) **10.1** (10.6)

A response pattern analysis—*In English*

Rep. Statist. Appl. Res. (**JUSE**) (1959) **6**, 82-96 (10 references, 4 tables)

The author presents three methods of singling out the true response pattern to one arrival of the meteorological stressor with or without noise. In fact, he gives the methods to solve the single response pattern $f(t)$ from the functional equation $F(t) = f(t-a)+f(t-b)$ under the condition that $F(t)$, a and b ($a<b$) are known and $f(t)$ is one valued bounded function of finite range. He shows that the temporal or spatial n-method which is widely used in this field of research to find out the response pattern to the meteorological stressors is generally biased. In sections 3, 4, 5, 6, 7, he considers three methods to solve the simplest case in which only two signals arrive at $t = a$ and $t = b\,(>a)$. In section 8, he extends the above idea to the two-dimensional case

$$F(t_1, t_2) = f(t_1-a_1, t_2-a_2)+f(t_1-b_1, t_2-b_2).$$

Furthermore, in section 9, he treats the case which has many signals in one dimension. (See also No. **2**/406).

(S. Sakino)

2/405

MASUYAMA, M. (Tokyo University) **10.1** (0.4)

Further studies on response pattern analysis—*In English*

Rep. Statist. Appl. Res. (**JUSE**) (1959) **6**, 97-99 (5 references)

In this paper, the author tries to solve by an iterative method the difference equation with constant coefficients

$$c_0\,Y(t)+c_1\,Y(t-1)+c_2\,Y(t-2)+\ldots+c_{k-1}\,Y(t-k+1) + Y(t-k) = G(t)$$

where $c_aY(t-a)$ is a one-valued bounded response pattern of finite range to the single signal or stressor of intensity c_a at $t = a$ and $G(t)$ is the observed response, and where the Laplace transform of $Y(t)$ is assumed to exist. (See also No. **2**/405).

(S. Sakino)

2/406

NEWELL, C. F. (Brown University, Providence, R.I.) **10.4** (10.1)

Queues for a fixed-cycle traffic light—*In English*

Ann. Math. Statist. (1960) **31**, 589-597 (8 references, 1 figure)

Analytic expressions are derived for the average queue length at a fixed-cycle traffic light, under equilibrium conditions.

The model is a discrete one, in which arrivals and departures of cars at the traffic light occur only on a set of equally spaced time-points. Each cycle of the periodic light consists of r consecutive " red " time-points followed by g " green " points. At each point one car may arrive (probability α) or cars not arrive (probability $1-\alpha$); independently from point to point. Departures occur at green points only, one departure for each time-point so long as there is a car present; with no queue, a car may arrive and depart at the same green point.

Queue lengths at corresponding time-points of successive cycles form a stationary Markoff chain. If q_x denotes the queue length just before the first red point of the xth cycle, and u_x the total number of arrivals during the xth cycle, the q_x satisfy the recursion relation

$$q_{x+1} = \max\{q_x+u_x-g, 0\}.$$

This relation is used in Section 3 to find approximations to $\mathscr{E}(q_x)$ where the arrival rate is low. Starting with an arbitrary distribution for q_1, the sequence of distributions for q_2, q_3, etc., converges to the equilibrium distribution if the average number of arrivals per cycle is less than the maximum departure rate, that is, if $\alpha(r+g)<g$. The rate of convergence is very high if $g-\alpha(r+g)>[\alpha(1-\alpha)(r+g)]^{\frac{1}{2}}$. An adequate approximation to the distribution of q_x may be obtained by finding the distribution of q_n for some small n. Approximations for $\mathscr{E}(q_x)$ are found in the limit r and $g\to\infty$, with r/g fixed, for the case $\mu>1$, where

$$\mu = [g-\alpha(r+g)][rg/(r+g)]^{-\frac{1}{2}}.$$

In Section 4 use is made of generating functions to handle the case $\mu<1$. The generating function is first shown to be the quotient of two polynomials of known degrees, and these polynomials are in turn shown to be determined by the roots of $z^g-(1-\alpha+\alpha z)^{r+g}=0$. For the case μ approximately equal to unity, a discussion is given in Section 5. The paper ends with a comparison of results with some previously obtained by Webster " Traffic signal settings " [Road Research Tech. Paper No. 39 (1958) Her Majesty's Stationery Office, London].

2/407

(R. Hooke)

PEARCE, S. C. (East Malling Res. Sta., Maidstone, England) **10.9** (6.0)

A method of studying manner of growth—*In English*

Biometrics (1960) **16**, 1-6 (1 table, 1 figure)

Questions about the growth rates of individuals in a growing organism can sometimes be answered by noting the relationship between the standard error of log(size) and the mean increment in log(size) of the organism. If $x_i(t)$ denotes the size of the ith individual at time t then the following three standard errors are especially useful in exploring the laws of growth:

(i) σ_t, that of log(size) at time t, that is to say of $L_{it} = \log x_i(t)$,

(ii) σ_{t0}, that of the increment in log(size) between time 0 and time t, that is of $L_{it}-L_{i0}$,

(iii) σ_t, that of log(size) at time t after adjustment by covariance on the corresponding values at time 0, that is of $L_{it}-\rho_{0t}\dfrac{\sigma_t}{\sigma_0}L_{i0}$,

where ρ_{0t} is the correlation between L_{it} and L_{i0}.

The author shows that only two of these are independent.

A study of graphs of the three standard errors when plotted against the mean difference between L_{it} and L_{i0} sometimes suggests laws of growth. For example

(i) if $\sigma_t = \sigma_t'$ from some time t onward it implies that the initial size ceases to affect the growth rates,

(ii) if the curve for σ_{t0} continues to rise it means that the growth rates for different individuals vary and,

(iii) if the curves for σ_t and σ_t' are not straight it implies that the growth rates are not constant.

These points are illustrated by two examples, the second of which deals with the growth rates of apple trees: graphs are given to illustrate the trials.

(R. L. Chaddha)

RUNNENBURG, J. T. (Mathematical Centre, Amsterdam) **10.4** (10.1)

Probabilistic interpretation of some formulae in queueing theory—*In English*

Bull. Int. Statist. Inst. (1960) **37**, III 405-414 (13 references)

It is customary to regard the variable z of a generating function $A(z) = \sum_{n=0}^{\infty} p_n z^n$ as "having no significance" [see Feller, *Introduction to probability theory and its applications* (1950) chapter 11 New York: Wiley], even if the quantities p_n generated are probabilities. An outstanding feature of Van Dantzig's "Method of collective marks" is that it recognises the possibility of inventing a "significance" for the variable z. Instead of using $A(z)$ as a receptacle for probabilities which is devoid of intuitive meaning with respect to the problem in hand, one may try to interpret $A(z)$ itself as a probability in a slightly more complicated problem than the original; in this z denotes a probability, which may have any value between 0 and 1.

It is shown by the paper, that the foregoing outline provides a method which has distinct advantages over other approaches in certain waiting-time problems. Three examples are dealt with; all originating from the one-server queueing theory with first-come, first-served queue discipline, in which the arrival intervals are exponentially distributed, while the service times have an arbitrary distribution

The examples are as follows:

(i) Example A treats a formula due to Takács [*Acta Math. Acad. Sci. Hung.* (1955) **6**, 101-129], concerning the virtual waiting-time at an arbitrary moment.

(ii) Example B gives a new derivation of the well-known formula for the Laplace-transform of the waiting-time distribution in the stationary situation, due to Pollaczek [see page 156 of Kendall, D. G., *J. R. Statist. Soc.* B (1951) **13**, 151-185].

(iii) Example C supplies intuitive content to a decomposition of the Pollaczek formula [see first footnote on page 208 of Kendall, D. G., *J. R. Statist. Soc.* B (1957) **19**, 207-233].

(J. T. Runnenburg)

2/409

SACKS, J. (Columbia University, N.Y.) **10.4** (10.1)

Ergodicity of queues in series—*In English*

Ann. Math. Statist. (1960) **31**, 579-588 (5 references)

Two queues are said to be in series if the output of the first queue is the input of the second queue. If W_n is the waiting time of the nth individual in the first queue, and W_n^* his waiting time in the second queue, then the queueing system is said to be ergodic if the joint distribution of (W_n, W_n^*) converges to a probability distribution as n becomes large.

In this paper, the conditions under which queues in series will be ergodic are derived; firstly for the case of two servers and then in the general case of s servers. The first significant result in the development is:

$$\Pr\{Z_n \leqq t \mid Z_1 = 0\} \rightarrow F(t) \text{ as } n \rightarrow \infty$$

where F is a two-dimensional distribution function whose variation over two-dimensional space may be less than one, that is, F may not be a probability distribution function. In the above, $Z_n = (W_n, W_n^*)$.

The second part of the argument is to show that under appropriate conditions, Z_n is bounded in probability as n becomes large, so that F will be a probability distribution. This is contained in a theorem: if $\mathscr{E}(g_2) > \max[\mathscr{E}(R_1), \mathscr{E}(p_1)]$ then F is a probability distribution. Here R_n is the service time of individual n at the first server; p_n is his service time at the second server; and $g_{n+1} = \tau_{n+1} - \tau_n$, where τ_n is the time at which individual n enters the system. It is assumed that each of the three sequences $\{R_n;\ n \geqq 1\}$, $\{g_n;\ n \geqq 2\}$, $\{p_n;\ n \geqq 1\}$ is a sequence of independent and identically distributed random variables, and that the three sequences are mutually independent.

The necessity of the condition of the above theorem is shown, when first moments are assumed to exist, by the following:

(i) If $\mathscr{E}(R_1) \geqq \mathscr{E}(g_2)$, then $F(t) = 0$,

(ii) If $\mathscr{E}(p_1) \geqq \mathscr{E}(g_2)$, then $F(t) = 0$.

The question of ergodicity in the case of s servers is handled in much the same way. Here let R_n^σ be the service time of individual n in the server σ, $\sigma = 1, 2, \ldots, s$. Let $R_{n+1}^0 = \tau_{n+1} - \tau_n$, and let $\mu_\sigma = \mathscr{E}(R_i^\sigma)$, $\sigma = 0, 1, 2, \ldots, s$. Then the analog to the theorem is: if $\max_{1 \leqq \sigma \leqq s} \mu_\sigma < \mu_0$ then F is a probability distribution: where F is the s-dimensional analog of F in the first displayed expression. Finally, if $\max_{1 \leqq \sigma \leqq s} \mu_\sigma \geqq \mu_0$ then F is identically zero.

(R. M. Durstine)

SCOROCHOD, A. V. **10.1** (–.–)

On differentiability of measures which correspond to Markoff processes—*In Russian*

Teor. Veroyat. Primen. (1960) **5**, 45-53 (5 references)

The conditions for the differentiability of measures that correspond to Markoff processes are investigated and the density of one measure with respect to the other is calculated.

(A. V. Scorochod)

2/411

SHISKIN, J. (Bureau of the Census, Washington, D.C.) **10.0** (11.5)

Decomposition of economic time series by electronic computers—*In German*

Math. Tech. Wirtschaft. (1960) **3**, 112-118 (5 references, 4 tables, 3 figures)

The article is a German translation of Shiskin's paper in *Science* (Vol. **128**, pages 1539-1546). That paper, in turn, is a summary of several articles by Shiskin & Eisenpress on this topic: the use of computers in the decomposition of economic time series [see, for example, *J. Amer. Statist. Ass.* (1957) **52**, 415-449]. These articles reflect the pioneer work carried out by Shiskin in this field during the last few years.

This article in *Science*, as translated in *Math. Tech. Wirtschaft* first discusses the different kinds of economic fluctuations; the author next goes into the possibilities of applying computers in the decomposition of economic time series, into the trend (including cyclical components) and into seasonal and irregular components.

In this paper, no details as to the method itself are given. The details of the rather elaborate and smooth mechanical method of decomposition as developed by Shiskin are set down in the other papers mentioned above. Broad emphasis, however, is given to tests of the " goodness of fit " of the method. Of the 10,000 series so far decomposed by the method, 150 leading series were carefully tested in several ways to allow a more accurate evaluation of this mechanical method of decomposition.

(G. Bruckmann)

2/412

SIEGEL, S. (Pennsylvania State University) **10.1** (2.5)

Theoretical models of choice and strategy behaviour: stable behaviour in the two-choice uncertain outcome situation—*In English*

Psychometrika (1959) **24**, 303-316 (22 references, 1 figure)

The author points out that in the classical experimental situation, as first used by Humphreys, in studying two-choice uncertain outcome situations two widely used theoretical models yield somewhat different predictions. The Estes statistical learning model yields a prediction that subjects will learn to match their response ratios to the actual probabilities of occurrence of the events. The von Neumann and Morgenstern game-theoretic model would predict that a person will learn to maximise the expected frequency of correct predictions. To do this he will predict the more frequent event on all trials.

The author purposes a resolution to the apparent paradox, a resolution which incorporates the utility-theory approach. More specifically, the author is suggesting that one should consider the utility of the person making a correct prediction of a given event. Three models are presented. Model III is of particular interest in that it incorporates the Shannon information measure and leads to somewhat different predictions than are yielded by either Models I or II.

An experiment is presented designed to test the hypothesis that asymptotic probability of a person's predicting the occurrence of the more frequent event in a two-choice uncertain outcome situation is a function of the level of reinforcement present in the situation. It is hypothesised that the probability of predicting the more frequent event will tend toward unity as the rewards (positive utility) and costs (negative utility) of correct and incorrect predictions are increased. The data presented confirms the prediction.

(R. E. Stoltz)

2/413

SMITH, C. (Animal Breeding Res. Org., Edinburgh) **10.9** (6.1)

Efficiency of animal testing schemes—*In English*

Biometrics (1960) **16**, 408-415 (3 references)

Testing schemes or procedures are used in the improvement of livestock. The amount of improvement obtained from a testing procedure depends on (*a*) optimum design with respect to predicted genetic gain and (*b*) optimum application to the whole population. Robertson [*Biometrics* (1957) **13**, 442-450] derived an expression for optimum design such that superiority of chosen families would be maximised.

The author has considered three cases in which individuals, rather than families, are selected, and has derived prediction formulas for genetic gain when

(i) only non-tested members of selected families are used for breeding,

(ii) all members of selected families are used,

(iii) only the tested animals are available.

Variables in these formulas include the proportion p of groups selected, the genetic relationship r between members of the same group, the observed intra-class correlation t for the character considered, the heritability ratio h^2, the genetic standard deviation σ_G, and the ratio K of number of animals tested to number of groups chosen. Expected genetic gains, using a wide series of testing situations for pigs, were calculated and are presented graphically.

Genetic gain showed a nearly linear relationship to the logarithm of the fraction to be selected from the groups tested. Increases in predicted genetic gain were associated with decreases in t, also with increases in size of family. Full-sib families were preferable if the fraction to be selected was small; half-sib families if the fraction was large. Selection of tested animals alone was advantageous if the fraction to be selected was small. The use of all members of tested families gave greater genetic gains than the use of only non-tested members.

Selection based on family mean plus individual merit should add, in practice, very little to genetic improvement from selection on family mean alone. If selection is to be done simultaneously for two or more traits, it is appropriate to use a selection index.

The individuals selected through the foregoing procedures may be used directly to improve the whole population, or else used indirectly through a nucleus of breeding stock. The nucleus plan, though at first inferior, should ultimately lead to more genetic improvement in the whole population than would the use of selected individuals directly on the whole population.

(N. R. Thompson)

2/414

STRATONOVICH, R. L. (Moscow) **10.1** (-.-)

Conditional Markoff processes—*In Russian*

Teor. Veroyat. Primen (1960) **5**, 172-195 (4 references)

In this paper the author gives relationships between the probabilities of conditional Markoff chains for neighbouring tests. The conditional probabilities at the end of the observation interval, the final probabilities, are satisfied with first-type equations corresponding to an increase in the observation interval. The second-type equations for the conditional probabilities within the observation interval are written in terms of the final probabilities.

The following special cases are considered: Gaussian noise with independent values which becomes a delta-correlational process when the moments of time are compacted, and a continuous Markoff process.

The related problem of the time-sign reversal of ordinary, *a priori*, Markoff processes is treated.

(R. L. Stratonovich)

2/415

VINCZE, I. (Math. Inst., Hungarian Academy of Sciences, Budapest) **10.5** (6.4)

On an interpretation of a concept of information-theory—*In Hungarian*

Mat. Lapok (1959) **10**, 255-266 (10 references)

The I-divergence of a random variable ξ, with respect to a probability distribution ϕ, is the expression

$$I_\phi(\xi) = \int_{-\infty}^{\infty} f(x) \log [f(x)/\phi(x)] dx$$

where $f(x)$ and $\phi(x)$ are the density functions of the variate ξ and the distribution ϕ respectively. In this paper the author gives the following interpretation of this notion: the probability distribution ϕ is interpreted as the distribution of our interest and $I_\phi(\xi)$ as the information contained in the value of ξ with respect to the distribution of our interest. This interpretation is made plausible by starting from Shannon's formula for the discrete case and passing to the limit.

In case ξ and ϕ are normal with parameters μ_0, σ_0 and μ, σ_0 respectively $I_\phi(\xi) = (\mu - \mu_0)^2/(2\sigma_0^2)$.

(K. Sarkadi)

2/416

VOGEL, W. (Universität Tübingen, Germany) **10.1** (5.7)

A random-walk problem and its application in the theory of sequential design—*In German*

Arch. Math. (1960) **11**, 310-320 (9 references)

The following random-walk problem is treated: beginning at the time $n = 0$ in the point $k = 0$, the random variable may jump at the times $n = 1, 2, 3, \ldots$, one unit in a positive or a negative direction; that is from k to $k+1$ or to $k-1$. The probability of a positive jump is p if k is positive, $1-q$ if k is negative, and $r = (p+1-q)/2$ if $k = 0$. The corresponding values for a negative jump are: $1-p$, q, and $s = (q+1-p)/2$. The case $p \neq 1-q$ is of especial interest.

If U_n is the position k of the random variable at time n, the author discusses the asymptotic behaviour of the following expression:

$$S_n(p, q) = \sum_{i=0}^{n} (\Pr(U_i<0)+\tfrac{1}{2}\Pr(U_i=0)),$$

that is, the probability that the position of the random variable will be negative at some of the times $1, \ldots, n$, and half the probability that it will be zero at some of these times. He proves several propositions on the asymptotic behaviour of this expression.

These results can be applied to the following sequential design: given two possibilities to perform an experiment, the outcome of which takes the values 1 (success) and 0 only. The two possibilities differ in the probabilities of success which may be denoted by p and q respectively. A total of n experiments are to be performed according to the following sequential plan: the first possibility must be used, if in the former experiments the number of successes with the first possibility plus the number of failures with the second one is greater than the number of failures with the first plus successes with the second possibility. If however the second number exceeds the first one, the second possibility must be used; and if the two numbers are equal, either of the two possibilities must be used with probability one-half. If $\mathscr{E}(V_i)$ is the expectation of the ith experiment, the risk function, by which the value of this sequential plan may be judged, can be written as $R(p, q) = n \max(p, q) - \sum_{i=1}^{n} \mathscr{E}(V_i)$. One can easily show, that this risk function is linearly connected with the expression $S_n(p, q)$ introduced in connection with the random walk discussed above, so that the results on asymptotic properties of S_n can be applied to this risk function.

2/417

(B. Schneider)

YUSHKEVIČ, A. A. (Moscow) **10.1** (-.-)

On the definition of a strong Markoff process—*In Russian*

Teor. Veroyat. Primen. (1960) **5**, 237-243 (3 references)

The equivalence of the two definitions for strong Markoff Processes given in Dynkin's book [*The basis of the theory of Markoff processes* (1959), Moscow] is proved.

(A. A. Yushkevič)

ASKOVITZ, S. I. (Univ. Pennsylvania, Philadelphia and Albert Einstein Medical Center) **11.3** (6.1)

Graphic methods based upon properties of advancing centroids—*In English*

J. Amer. Statist. Ass. (1959) **54**, 668-673 (8 references, 6 figures)

In an earlier paper [*J. Amer. Statist. Ass.* (1957) **52**, 13-17] the author presented a graphical method for obtaining the best fitting, i.e. least squares straight line to n points having equally spaced abscissae. In the present paper he gives: (i) two alternative solutions to the same problem; (ii) a construction for three unequally spaced points; (iii) a method which uses the best fitting straight line to n consecutive equally spaced points to obtain the best line given one additional point; (iv) a construction which locates the mean of a histogram. Proofs are given, or at least sketched.

(i) The first alternative consists in repeated addition of new points as described under (iii) below. The second consists in starting from the second point from the left (rather than the first, as described in the earlier paper) and obtaining at the $(n-2)$th step the weighted centroid of the n points, with weights $0, 1, \ldots, n-1$; rather than $1, 2, \ldots, n$. Repeating with left and right interchanged gives another weighted centroid, which, with the first, determines the least squares line.

(ii) Given unequally spaced points A, B, C, let A_0, B_0, C_0 be their projections on the x axis. Construct A_1, B_1, C_1 alternately above and below A, B, C, so that $AA_1 = B_0C_0$, $BB_1 = C_0A_0$, and $CC_1 = A_0B_0$. Then the two intersection points of the two polygonal lines ABC and $A_1B_1C_1$ determine the least squares line.

(iii) Let P and Q be the trisection points of the given best fitting line terminated at the first and nth abscissae, and let F be the additional point. Locate R on QF two-thirds of a spacing unit, horizontally to the right of Q. Then P and R determine the new best fitting line.

(iv) The histogram is replaced by points centred on the tops of the bars. The mean is located by a method which involves the two weighted centroids described under (i).

(R. J. Buehler)

2/419

BANERJEE, K. S. (State Statistical Bureau, Calcutta) **11.8** (-.-)

A generalisation of Stuvel's index number formulae—*In English*

Econometrica (1960) **27**, 676-678

The purpose of this note is to generalise the derivation of the indexes given by Stuvel in his recent article, " New index number formulae " [*Econometrica* (1957) **25**, 123-131].

After discussing Stuvel's procedure for deriving the formulae, the author presents his generalisation which satisfies both the time-reversal and the factor-reversal tests. The basic identities, of which Stuvel took the arithmetic average, are now averaged with weights which incorporate an alternating function of price and quantity.

(W. R. Buckland)

2/420

FITZPATRICK, P. J. (Catholic Univ. of America, Washington, D.C.) **11.9** (-.-)
Leading British statisticians of the nineteenth century—*In English*
J. Amer. Statist. Ass. (1960) **55**, 38-70 (59 references)

The author reviews the lives and statistical contributions of eleven outstanding British statisticians of the nineteenth century, namely, Playfair, Porter, Babbage, Farr, Guy, Newmarch, Jevons, Rawson, Galton, Giffin, and Edgeworth. He states that perhaps the most important contributions are those of Galton, who introduced the methods of correlation and regression analysis, the latter arising out of his studies on the relationship between the stature of an adult person and the average stature of the parents.

In the conclusion, the author summarises and compares the training and professional activities of the men, the offices which they held, and the honours conferred upon them, together with their contributions to statistics.

(S. P. H. Mandel)

2/421

GOLENKO, D. I. & SMIRYAGIN, V. P. (Computing Centre, Acad. Sci. USSR, Moscow) **11.7** (11.6)
A source of random numbers which are equi-distributed in [0, 1]—*In Russian*
Publ. Math. Inst. Hung. Acad. Sci. (1960) **5**, 241-253 (2 references, 11 figures)

In this paper the authors deal with an electronic device yielding random binary numbers; it makes use of occurrences of noise voltage which are greater or less than some fixed voltage level. The background from which these random numbers arise is of fully stochastic nature; thus, they might have some advantage over the pseudo-random numbers generated by any of the usual methods. The randomness of the obtained random numbers has been checked by several statistical tests. The results were satisfying; thus the device can be applied, for instance, in solving problems by the Monte Carlo method.

(P. Medgyessy)

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KENDALL, M. G. (London School of Economics) **11.9** (-.-)

Studies in the history of probability and statistics. X. Where should the history of statistics begin?—*In English*

Biometrika (1960) **47**, 447-448

The author is concerned to point out that the isolated enumerations and inventories of the first sixteen hundred years of this present era do not constitute " statistics " in the sense that the word is now used. Statistics as we understand it only really began when the collectors of data began to reason about it and to seek to draw inferences which were not the immediate result of the straightforward tabulation of the data. The random element has been studied from earliest times. The collection of data such as that of the Book of Numbers or the Domesday Book is also a process of some antiquity, but as the author remarks, the marriage of the random element to the collection of data had to wait until well on into the nineteenth century.

Kendall dates the real beginning of " statistics " with the publication by John Graunt of *Observations on Bills of Mortality* in 1660 A.D.

(Florence N. David)

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LOCKS, M. O. (University of California, Los Angeles) **11.5** (7.1)

Automatic programming for automatic computers—*In English*

J. Amer. Statist. Ass. (1959) **54**, 744-754 (6 references, 2 tables, 2 charts)

A description of the various stages of work and their attendant difficulties and complications when compiling programmes for automatic computers is the main concern of the author in the first part of this paper. Having described the advantages of automatic programmes, these include simplified methodology and the time saved by the repeated use of the same programme, he describes in detail an automatic programming system and enumerates the three basic types: namely the interpretative, assembly and compiling systems; these are briefly described and their merits compared. The availability of these three types for different computers is discussed: the question of excessively long running time and poor adaptability to many types of problems is then dealt with briefly and one method of solving this problem is given. An example is given of an automatically programmed analysis of variance F computation using the Math-Matic compiler system for the Univac I and II.

Hypothetical costs are estimated for computing the Snedecor F in three different ways. These are as follows:

(i) Desk calculator

(ii) Computer with hand coding but without automatic programming

(iii) Computer with automatic coding and employing a compiler.

Having stated that, for simplicity, linear cost functions will be assumed the remainder of this part of the paper gives the various costs and compares them: two tables and a chart are used to illustrate this section.

The final part of this paper deals with the analysis of the results and there is a summary which includes prognostications as to the future of automatic programming and states that anticipated future developments include automatic operating and debugging.

(A. Booker)

MACKINNON, W. J. (University of Arizona, Tuscon) **11.1** (2.4)

Compact table of twelve probability levels of the symmetric binomial cumulative distribution for sample sizes to 1000—*In English*

J. Amer. Statist. Ass. (1959) **54**, 164-172 (15 references, 1 table)

A table of critical values for tests of the symmetric ($p = \frac{1}{2}$) binomial cumulative distribution is presented. Compactness of the table obtained is accomplished by the use of an unusual table argument, d, defined as follows: let s and r be the frequencies of " successes " and of " failures " in n trials such that $s \leqq \frac{1}{2}n$ and $r \geqq \frac{1}{2}n$; then the sum, $r+s$, is n and the difference, $r-s$, is d. The table gives the critical values of s at twelve two-tail probability levels (0·001, 0·01, 0·02, 0·05, 0·10, 0·20, 0·30, 0·50, 0·70, 0·80, 0·90, and 0·95) for sample sizes to 1000. The entire table requires only those values of d from 1 through 105, thus the use of d rather than n as the argument reduces the size of the table considerably.

Since the table has limits on both sample sizes and probability levels, the author discusses other techniques for making tests involving the symmetric binomial distribution. In addition to giving references to some available tables, he gives the maximum error of the two-tail probability associated with both the normal approximation and the Camp-Paulson approximation when $p = \frac{1}{2}$. It is noted that when $n \geqq 5$, the normal approximation has a smaller maximum error than the Camp-Paulson approximation.

The justification of the validity of a table using d as the argument depends upon the monotonicity of the cumulative probability, with respect to s, for fixed d. A proof of this is given as well as an explanation of the basis for the rules which appear in the instructions for the use of the table. Finally, the methods used to obtain the table entries are given.

(D. D. Grosvenor)

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PERKAL, J. (University of Wrocław) **11.0** (9.2)

On the analysis of a set of characteristics—*In English*

Zast. Mat. (1960) **5**, 35-45

After a short critical review of the advantages and disadvantages of known statistical methods used in analysis of factors (Hotelling, Spearman, Thurstone) the author proposes two new methods [Perkal, J., " Two new methods of analysing a collection of attributes " *Bull. Acad. Sci.*, III (1959) 7, 63-66: abstracted in this journal No. **1**/324, **11.0**] which can be summarised as follows:

let x_{ij} be the value of the ith attribute of the jth individual, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, n$, and let the attributes be standardised over the set of individuals to the effect of having zero mean and unit variance.

If all the attributes are non-negatively correlated, then $m_j = \frac{1}{k} \sum_i x_{ij}$ is the value of a common factor for the jth individual, while the residuals or indices $w_{ij} = x_{ij} - m_j$ contain the remaining part of information. In the contrary case a procedure for splitting the collection of attributes into groups of non-negatively correlated attributes is defined, which is based on the matrix of correlation coefficients between the attributes. The summarising factors are then computed for each group of attributes separately. A k-dimensional vector with all its components equal to m_j in the first case and with equal components for each group of attributes in the second case is assumed as the representation of the common factor; and this procedure of computing it is called by the author the " single-vector " method.

A " multiple-vector " method is also defined. It consists in successive application of the above described single-vector method to the ever smaller groups of indices, which, after suitable standardisation are regarded as new collections of attributes. The procedure stops when there are no more groups of non-negatively correlated indices.

In the way described above the factors are comparatively easily and uniquely determined and may be immediately interpreted in natural sciences, although, unlike in Spearman's and Thurstone's methods, they must not be mutually orthogonal and uncorrelated with the residuals which appear when the computations are continued.

(S. Zubrzycki)

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